Chapter 10

Experiment 8: Electromagnetic Resonance

10.1 Introduction

If a pendulum whose angular frequency is $\omega_0$ for swinging freely is subjected to a small applied force $F(t) = F_0 \cos \omega t$ oscillating with a different angular frequency, $\omega$, the force acts during part of each cycle in a direction opposite to velocity of the pendulum, thereby partially negating its effect in accelerating the pendulum during the rest of the cycle. The strongest response to the applied force would be expected when $\omega \approx \omega_0$, since then the force stays in phase with the velocity of the pendulum and can act throughout the entire cycle in the same direction that the pendulum is moving. If frictional effects are small, a pendulum starting from rest can attain a large amplitude of oscillation from the cumulative effect of even a weak oscillating force acting over many periods of oscillation, provided the applied driving force is precisely tuned to the characteristic frequency of oscillation. Far more important for practical uses than the selective response of a pendulum to the frequency of an applied force, however, is the corresponding effect in an electric circuit.

We examine this kind of behavior for electromagnetic oscillations in the $RLC$ circuit, where the resistance, inductance, and capacitance are analogous, respectively, to friction, inertia (or mass), and a spring-like restoring force in mechanics. In the previous experiment, the square-wave generator repeatedly produced a steady voltage to charge the capacitor, followed by an abrupt change to zero; this allowed the circuit to oscillate freely at its natural frequency while the circuit resistance damped the motion to zero. These were observations of the circuit’s transient response. Since the natural circuit response is sinusoidal, it is reasonable to wonder how the circuit will respond to a sinusoidal $V(t) = V_0 \sin \omega t$

\[ V_L \quad I \quad V_C \quad V_R \]

\[ R \quad L \quad C \]

Figure 10.1: A schematic diagram of our $RLC$ circuit. This particular circuit has the four elements in series so all of the currents are the same.
stimulus. It turns out that when the stimulus is first applied or when the sinusoid is quickly changed, the circuit will always respond with its transient response until they naturally damp themselves out. Once the transients are gone, the remaining motion is the steady state response. This class of differential equation is not capable of any other kinds of solutions. The most general response of RLC circuits is

\[ V_C(t) = V_{\text{transient}}(t) + V_{\text{steadystate}}(t) \]  

(10.1)

In the present experiment, we connect another RLC circuit to a sine-wave generator as shown in Figure 10.1 and observe its response for different frequencies,

\[ V(t) = V_0 \sin \omega t \]  

(10.2)

This signal has amplitude, \( V_0 \), and angular frequency, \( \omega \). The circuit is called a “driven” RLC oscillator. We will see that the response will also be sinusoidal, so this is also a driven, damped harmonic oscillator. We seek to determine expressions for the resulting current, \( I(t) \), in such a circuit in what follows. We use Kirchhoff’s loop rule around the circuit,

\[ 0 = V(t) - V_L - V_C - V_R \]  

(10.3)

Since \( Q = CV_C \) for a capacitor, its current is

\[ I = \frac{dQ}{dt} = C \frac{dV_C}{dt}. \]  

(10.4)

The same current flows through all of the components so this makes

\[ V_R = IR = RC \frac{dV_C}{dt} \]  

(10.5)

where we have applied Ohm’s law to the resistor and

\[ V_L = L \frac{dI}{dt} = LC \frac{d^2V_C}{dt^2} \]  

(10.6)

for the inductor. Thus we substitute from Equation (10.2) for \( V(t) \) and rearrange Equation (10.3) to find that

\[ LC \frac{d^2V_C}{dt^2} + RC \frac{dV_C}{dt} + V_C(t) = V_0 \sin \omega t \]  

(10.7)

is the differential equation that we must solve to predict the circuit’s response. This inhomogeneous second order differential equation with constant coefficients is much harder to solve than the homogeneous one from last week; however, the mathematicians have managed it. This differential equation in standard form can be written

\[ \frac{1}{\omega_0^2} \frac{d^2f}{dt^2} + \frac{1}{Q\omega_0} \frac{df}{dt} + f(t) = f_0 \sin \omega t \]  

(10.8)
CHAPTER 10: EXPERIMENT 8

where this quality factor, \( Q \), is not the capacitor (or any other) charge. In the Appendix we demonstrate a strategy to use ‘phasors’ in the solution of these complicated equations. Last week we defined the ‘resonance frequency’ of the \( RLC \) circuit to be

\[
f_0 = \frac{1}{2\pi\sqrt{LC}} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}}. \tag{10.9}
\]

This week we also will need the quality factor at resonance

\[
Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}. \tag{10.10}
\]

If \( Q \) is small, the circuit responds to a very wide range of frequencies; if \( Q \) is large, the circuit responds to a very narrow range of frequencies. This property of resonance leads us to the concept of “bandwidth”,

\[
\Delta \omega = \frac{\omega_0}{Q} \quad \text{or} \quad Q = \frac{f_0}{\Delta f}. \tag{10.11}
\]

The \( LRC \) circuit is one example of a “filter”. Signals with \( \omega_0 - \Delta \omega \leq \omega \leq \omega_0 + \Delta \omega \) are passed to the output but other frequencies are blocked. \( \Delta \omega \) is the width of the “passband” or the bandwidth of the filter. Strictly speaking this bandwidth is a little ‘fuzzy’; frequencies barely inside the passband are attenuated noticeably and frequencies barely outside the passband are passed somewhat. The edges of the passband are the frequencies where the passed power is half of its maximum.

It turns out that we can manufacture variable inductors and capacitors whose values change as we rotate a knob. By using one of these in our filter, we can vary the resonance frequency along a certain range. Increasing \( L \) also increases the quality factor and decreases the bandwidth. Increasing \( C \) decreases the resonance frequency but does not change the bandwidth.

**Checkpoint**

Can you think of a use for a circuit whose resonance frequency can be varied over the frequency range \( 88.1 \text{MHz} \leq f \leq 108.1 \text{MHz} \)?

Equation (10.10) gives the quality factor only at resonance but we also define a capacitive
and inductive $Q$ at all frequencies,

$$Q_C = \frac{1}{\omega RC} \quad \text{and} \quad Q_L = \frac{\omega L}{R}. \quad (10.12)$$

The quality factor at resonance of macroscopic mechanical systems is limited to about 100 due to the prevalence of friction. The quality factors of electromagnetic systems can be $10^4$ to $10^6$. The quality factor of atomic systems can be $10^{10}$. Recently, the quality factor of a coherent laser with a cooled coherent atomic gas amplifying medium reached a new record of $10^{17}$. Resonances with large $Q$ can be used to make very precise and accurate clocks.

**Checkpoint**

What characterizes a resonance?

**Checkpoint**

What is meant by the “resonance frequency” of an $RLC$ circuit? For a swinging simple pendulum, what is the resonance frequency?

Since the power delivered to a resistor is $P = I^2 R$ and the band edge pass half of the power at resonance, we must have

$$I(\omega_{\pm}) = \frac{I_0}{\sqrt{2}}. \quad (10.13)$$

at each of the passband edges ($\omega_-$ is the lower edge and $\omega_+$ is the upper edge).

**Checkpoint**

What is the meaning of the $Q$ value of an oscillator? How does the $Q$ value change with increased resistance in the circuit? What feature of the response curve is described by the $Q$ value of the resonance?

At frequencies below resonance, the current peaks before the applied voltage. At frequencies above resonance, the current peaks after the applied voltage. Exactly at the passband edges the phase has shifted by $\varphi = \pm 45^\circ$.

We say the current “leads” the voltage for frequencies below resonance. We say the current “lags” the voltage for frequencies above resonance. The two particular cases having frequencies at the band edges are illustrated in Figure 10.3. As the applied frequency extends toward 0 Hz, the current wave at the top of the figure will move toward $\pi/2$ or $90^\circ$ phase shift. As the applied frequency extends toward higher frequencies (infinity) the wave at
the bottom of the figure will move toward $-\pi/2$ or $-90^\circ$ phase shift. This frequency dependence of phase shift is also illustrated in the bottom of Figure 10.2. In the series circuit that we have here, the largest voltage will develop across the component with the largest reactance; therefore, the largest reactance will determine the phase shift. Since

$$X_L = \omega L \quad \text{and} \quad X_C = \frac{1}{\omega C},$$  \hfill (10.14)$$
the inductor will have the largest reactance at high frequencies and the capacitor will have the largest reactance at low frequencies. To help keep track of the phase relation for the different components, remember “ELI the ICE man”. When the inductor’s phase relation is desired, ELI reminds us that for $L$, $E$ is before $I$ where the integral of electric field is voltage; voltage leads current. When the capacitor’s phase relation is desired, ICE reminds us that for $C$, $I$ is before $E$; current leads voltage. With regards to the series $RLC$ circuit, $X_C$ is larger below resonance so ICE reminds us that current leads voltage; $X_L$ is larger above resonance so ELI reminds us that voltage leads current or current lags voltage.

**Checkpoint**

What does “the current leads the voltage” mean? Give an example of when this occurs.

### 10.2 The Experiment

In this experiment we will observe the frequency dependence of the amplitude and phase shift of the current in a series $RLC$ experiment near resonance. Our inductor, capacitors, and function generator will be the same as last week, but this week we will install a $47\, \Omega$ resistor to allow us to monitor the current on an oscilloscope (a voltage sensing device). Since Ohm’s law gives us $I = \frac{V_R}{R}$, we can divide the resistor’s voltage by $47\, \Omega$ to get the current to be the same function of time as the applied voltage. Since there are no derivatives in this expression, the phase of the resistor’s voltage and current are equal; only the magnitudes are different and these scale as shown by the resistance itself. We will use Pasco’s Voltage Sensors and 850 Universal Interface for the personal computer and their Capstone program to emulate an oscilloscope.
10.3 Procedure

Wire up the circuit as shown in Figure 10.4. Verify that this implementation is indeed represented schematically by Figure 10.1. Be sure to short the black ground leads of the voltage sensors and generator together as shown so that one or more of them do not short out part of your circuit. Download the Pasco Capstone configuration file from the lab website,

http://groups.physics.northwestern.edu/lab/em-oscillations2.html,

and execute it. You can probably just click the link and then allow Capstone to load it directly. Turn on the 850 Universal Interface by lightly holding the power button for a second. You can click the “Signal Generator” button displayed at Capstone’s left to display and to remove the signal generator controls. Turn on the signal generator and set the desired frequency; begin with 125 Hz or so. Click the “Monitor” button to see the stimulus and the response; the button changes to “Stop” so that clicking it again pauses the data acquisition and display.

Now watch the resistor voltage (current) on Channel B as you slowly increase the frequency of the function generator by 10 Hz increments. Continue until you notice a change in the waveforms; this should occur at frequency between 150-600 Hz. Note your observations in your Data. Watch the amplitude and the relative phase of the current. When you locate the resonance, change the increment to 1 Hz and observe the resonance in more detail. Measure the resonance frequency when the current is in phase with the applied voltage. You can change the increments to 0.1 Hz if necessary. Also, note your measurement uncertainty and units in your Data. Now we would like to measure the bandwidth.

Stop the scope at the bottom left. Select the current channel and get a Measurement Tool. Set the origin at the waveform’s valley or at the waveform’s peak. Record the resonance frequency and maximum response amplitude (either peak or peak-to-peak) in your data.
Data. What are the units and error for this measurement? Note that the signal generator’s amplitude is 6 V; the peak-to-peak value is 12 V.

Helpful Tip

Measuring the peak-to-peak voltage doubles your signal and prevents bias in your data from amplifier input offsets. Measuring the peak value requires only one measurement instead of two but probably will introduce an error source you should mention in your writeup. The choice is yours. If you choose peak-to-peak measurements, you can enable the Delta Tool.

Now we want to estimate the circuit’s ‘bandwidth.’ Divide the response amplitude, $V_0$, at resonance measured above by $\sqrt{2}$ as suggested by Equation 10.38. Note this value for use in the next two steps. Find the frequencies, $V(f_+) = \frac{V_0}{\sqrt{2}}$ and $V(f_-) = \frac{V_0}{\sqrt{2}}$, above and below resonance that each gives a resistor voltage $\frac{V_0}{\sqrt{2}} \pm 5\%$. Note that simultaneously the phase shifts are $\pm 45^\circ$. Record both of these frequencies and the response amplitudes (peak or peak-to-peak as measured above). Subtract these two frequencies to estimate the bandwidth, $\Delta f = f_+ - f_-$. Record your work in your Data. Use the measured resonance frequency and bandwidth to estimate the circuit’s quality factor as in Equation 10.36. You can use the three points you have measured above to begin your observation of the amplitude’s response function below.

10.3.1 Resonance Amplitude Response

Now fill out Table 10.1 and type the results into Vernier Software’s Ga3 graphical analysis program. You can also download a suitable Ga3 setup file,

http://groups.physics.northwestern.edu/lab/em-oscillations2.html.

Unfortunately, your browser will open this text file itself unless you save it to your data folder first. Double-click the “RLC Resonance.ga3” that you downloaded and type in your data from Table 10.1.

Helpful Tip

Regardless where you save RLC Resonance.ga3 (preferable in Documents\Students), you can execute it from Mozilla’s Firefox® browser by clicking the down-pointing blue arrow at the top right; click the file from the drop-down box. It might be wise to make yourself a folder and to save your data there as you go.

Fit your data to the Lorentzian line shape in Equation (10.35). Drag a box around your data and select the “Resonance Peak” function from the bottom of the list. You may verify
Table 10.1: A place to record observations of frequency versus generator and resistor voltage.

<table>
<thead>
<tr>
<th>Formula</th>
<th>Frequency (Hz)</th>
<th>$V_{\text{gen}}$ (V)</th>
<th>$V_r$ (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0 - 3 \Delta f$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 - \frac{\Delta f}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 - \frac{\Delta f}{8}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 + \frac{\Delta f}{8}$</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>$f_0 + \frac{\Delta f}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 + 3 \Delta f$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

the model to be

$$V_r / \sqrt{1 + (Q \cdot (x/f_0 - f_0/x)^2)}.$$

Drag the fit parameters box off of your data. If your fit parameters box does not include uncertainties in the fitting parameters, right-click the box, “Properties...”, and select the “Show Uncertainties” checkbox.

**Helpful Tip**

You can copy your data table and your graph directly to your Word processor report. Click on the object to copy, “Edit/Copy”, activate Word, and ctrl+v. While the object is selected in Word, “Insert/Caption...” to give it a label. You can also adjust the object’s properties to yield a nice layout in the document.

10.3.2 Measuring $R$, $L$, and $C$

The capacitances are specified by the manufacturer to 5% tolerance. Thus we can use Equation (10.9) and the measured resonance frequency, $f_0$, to determine the inductance

$$L = \frac{1}{(2 \pi f_0)^2 C}$$
Table 10.2: A place to record observation of frequency versus the time shift needed to find phase shift.

<table>
<thead>
<tr>
<th>Formula</th>
<th>Frequency (Hz)</th>
<th>$t_\phi$ (s)</th>
<th>$T$ (s)=1/f</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0 - 3\Delta f$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 - \frac{\Delta f}{2}$</td>
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<tr>
<td>$f_0 - \frac{\Delta f}{8}$</td>
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<tr>
<td>$f_0$</td>
<td></td>
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<tr>
<td>$f_0 + \frac{\Delta f}{8}$</td>
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<td>$f_0 + \frac{\Delta f}{2}$</td>
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<tr>
<td>$f_0 + 3\Delta f$</td>
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</tr>
</tbody>
</table>

where we have used $\omega = 2\pi f$

Next we can use this inductance, the quality factor, and Equation (10.10) to determine the total resistance

$$R = \frac{2\pi f_0 L}{Q}.$$  

Assuming the quality factor and resonance frequency have uncertainties much less than 5%, $L$ and $R$ each will also have errors about $5\% - \delta L = 0.05L$ and $\delta R = 0.05R$.

10.3.3 (option) Resonance Phase Response

IF you have time, use the Smart Tools to measure the phase shift versus frequency curve. Complete Table 10.2 and enter it into Ga3. Measure the time between where the generator voltage crosses zero and where the test resistor’s voltage crosses zero. If the resistor voltage comes first (is to the left) give the time a positive sign and if it comes after the generator voltage give it a negative sign. Now verify that the phase shift column computes its values using

$$\text{“Tdiff” * “Frequency” * 6.2832.} \quad (10.15)$$

Be sure your column names and units are correct. Paying attention to the phase versus frequency graph, drag a box around your data points, and “Analyze/Curve Fit...” to
Equation (10.32). “Resonance Phase” at the bottom of the equation list uses

\[
\text{atan}(Q \ast ((f_0^2 - x^2) / (f_0 \ast x)))
\]  

Equation (10.16) as its model.

Be sure the fit model passes through your data points before you OK and drag your fit parameters box off of your data. Are your f0 and Q the same as those you got from the other graph? The resonance frequency is much more accurate from this graph, so feedback systems are usually designed to lock to this characteristic of resonance. “File/Print...” and enter the number in your group into Copies to get graph and table printouts for each of your group’s notebooks. Also, copy and paste your table and graph into your Word document.

10.4 Analysis

Does the resonance response describe your data adequately? Part of this answer rests on the quality of the model fit(s) in your graph(s). Do the model parameters agree with the circuit parameters? Use the strategy in Section 2.9.1 to decide whether the resonance model has predicted the correct resistance and inductance.

What other subtle sources of error can you find in this experiment? Does the model result from any assumptions that might not be met? What affects magnetic flux through the inductor that we have ignored? Can you identify any likely energy losses other than circuit resistance? Are any of these other errors large enough to explain any disagreement(s)? How well does the Lorentzian distribution predicted by our circuit analysis fit our data? How about the phase shift data?

10.5 Conclusions

What equations are supported by our experiment? (If you numbered your equations, you can simply quote the references here.) Communicate using complete sentences and define all symbols. Did we measure anything worth reporting in our Conclusions? If so give their values, units, and errors. How might we improve the experiment? Have you noted potential applications for what you have observed?
10.6 Appendix: Phasors

Sinusoidal waveforms consist of amplitude, frequency, and constant phase, $A \sin(\omega t + \varphi)$. We might consider using Euler’s formula,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$  \hspace{1cm} (10.17)

to express sinusoidal waveforms using exponential notation. First, change the sign of $\theta$ in Equation (10.17) to find the complex conjugate relation,

$$e^{-i\theta} = \cos \theta - i \sin \theta.$$  \hspace{1cm} (10.18)

We will use “∗” to represent complex conjugation. Next, form the difference between Equation (10.17) and Equation (10.18),

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{\cos \theta + i \sin \theta - (\cos \theta - i \sin \theta)}{2i} = \sin \theta.$$  \hspace{1cm} (10.19)

Since the terms on the left are a complex conjugate pair, we can always derive one from the other by complex conjugation ($i^* \rightarrow -i$); we might as well just replace $\sin \theta \rightarrow e^{i\theta}$ with the understanding that subtracting the complex conjugate (c.c.) and dividing by $2i$ will return the sinusoid. We can then express the sinusoid above as

$$A \sin(\omega t + \varphi) \rightarrow Ae^{i(\omega t + \varphi)} = Ae^{i\varphi}e^{i\omega t} = ae^{i\omega t}$$  \hspace{1cm} (10.20)

where we have absorbed the real amplitude and constant phase into a complex phasor, $a = Ae^{i\varphi}$. Those familiar with trigonometric identities will immediately see how much easier it is to deal with the sum of two angles using exponential notation. In fact any complex number can be expressed as a phasor and this makes phasors valuable for representing impedance and current response in electronic circuits. (In the strictest sense, ‘phasor’ implies a relationship to a sinusoid having amplitude, $|a| = A$, and phase, $\arg(a) = \varphi$.)

Although we do not yet know the capacitor’s voltage as a function of time, we can observe it on an oscilloscope to be a sinusoid and to have the same frequency, $\omega$, as $V(t)$. Since any sinusoid is an amplitude, a frequency, and a phase, knowing the capacitor’s response frequency allows us to write down an expression for the voltage

$$V_C(t) = v_C e^{i\omega t}.$$  \hspace{1cm} (10.21)

This also allows us to write an expression for the circuit’s current

$$I(t) = C \frac{dV_C}{dt} = i \omega C v_C e^{i\omega t} = i e^{i\omega t}$$  \hspace{1cm} (10.22)

where so far $i$ and $v_C$ are unknown phasors. We must be very careful not to confuse the current’s phasor $i$ with the imaginary unit $i$. Electrical engineers have developed the convention of using ‘$j$’ for the imaginary unit to reduce the risk of mixing them up.
Equation (10.22) does tell us that we must have
\[ i = i \omega C v_C. \]  

(10.23)

### 10.6.1 Solution to the Motion

Now we are ready to put these phasors to work in order to solve Equation (10.8). I reproduce it here for your convenience
\[ V_0 \sin \omega t = LC \frac{d^2 V_C}{dt^2} + RC \frac{dV_C}{dt} + V_C(t). \]

First, we substitute Equation (10.21) into this to find
\[
\frac{dV_C}{dt} = i \omega v_C e^{i\omega t}
\]
\[
\frac{d^2 V_C}{dt^2} = (i \omega)^2 v_C e^{i\omega t} = -\omega^2 v_C e^{i\omega t}
\]
\[
V_0 e^{i\omega t} = -\omega^2 LC v_C e^{i\omega t} + i \omega RC v_C e^{i\omega t} + v_C e^{i\omega t}
\]
\[
V_0 = -\omega^2 LC v_C + i \omega RC v_C + v_C
\]

where we have canceled the common factor $e^{i\omega t}$. Next, we use Equation (10.23) to substitute for $v_C$ and get

\[
V_0 = -\omega^2 LC \frac{i}{i \omega C} + i \omega RC \frac{i}{i \omega C} + \frac{i}{i \omega C} = \left( R + i \left( \omega L - \frac{1}{\omega C} \right) \right) i = zi
\]

(10.24)

where we have defined the “complex impedance” $z = R + i \left( \omega L - \frac{1}{\omega C} \right)$.

Although it isn’t obvious from this development, $V_0 = V_0 e^{i0} = v_0$ is also a complex phasor; it simply happens to have zero phase. The above development works exactly the same, however, if the applied stimulus has nonzero phase. Observant students will have noticed the similarity between Equation (10.24) and Ohm’s ‘law’. In fact we call Equation (10.24) “Ohm’s Law for AC circuits”. Observant students might also have noticed that resistors simply contribute $z_R = R$ to the impedance and inductors and capacitors contribute

\[
z_L = i \omega L \quad \text{and} \quad z_C = \frac{1}{i \omega C};
\]

(10.25)

this is why we define the reactances of these components by Equations (10.14).

We can represent the impedance of Figure 10.1 using phasor notation,

\[
Z e^{i\varphi_z} = z = R + i \left( \omega L - \frac{1}{\omega C} \right)
\]

(10.26)
where the magnitude is
\[ Z = |z| = \sqrt{z^*z} = \sqrt{R^2 + \left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)^2} \] (10.27)
and phase satisfies
\[ \varphi_z = \arg(z) = \tan^{-1}\left(\frac{\Im(z)}{\Re(z)}\right) = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right). \] (10.28)

The phasor for the circuit current can be found easily by dividing the applied voltage’s phasor by the circuit’s impedance phasor,
\[ i = \frac{v_0}{z} = \frac{V_0}{Z} e^{i\varphi_z} = V_0 Z e^{-i\varphi_z}. \] (10.29)
We have used the fact that the applied voltage has zero phase so that its phasor is real (\(e^{i0} = 1\)) and equal to its magnitude. We must note immediately that the current’s phase has opposite sign to the impedance’s phase whenever the applied voltage’s phasor is real.

Now we can immediately write down the circuit’s current,
\[ I(t) = I_0 \sin(\omega t + \varphi_I) \] (10.30)
where
\[ I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + \left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)^2}} \] (10.31)
and
\[ \varphi_I = -\varphi_z = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right). \] (10.32)
We can put Equation (10.31) into the standard form. First we bring out the resistance,
\[ I_0 = \frac{\frac{V_0}{R}}{\sqrt{1 + \left(\frac{Q}{\omega_0}\right)^2 \left(\frac{\omega L - \frac{1}{\omega C}}{\omega_0}\right)^2}} = \frac{\frac{V_0}{R}}{\sqrt{1 + Q^2 \left(\frac{R}{\omega_0 L}\right)^2 \left(\frac{\omega L - \frac{1}{\omega C}}{\omega_0}\right)^2}} = \frac{\frac{V_0}{R}}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_0 L} - \frac{1}{\omega_0}\right)^2}} \] (10.33)
and use the definitions of the resonance frequency, \(\omega_0\), and quality factor, \(Q\). We can also put Equation (10.32) into the standard form by using the definitions of quality factor and resonance frequency,
\[ \varphi_I = \tan^{-1}\left(\frac{1}{\frac{\omega L - \frac{1}{\omega C}}{R}}\right) = \tan^{-1}\left(\frac{QR \frac{1}{\omega_0} - \omega L}{\frac{R}{\omega_0}}\right) = \tan^{-1}\left(\frac{Q}{\frac{1}{\omega_0 L} - \omega_0}\right) \]
\[ = \tan^{-1}\left(\frac{Q \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega}}{\omega_0}\right) = \tan^{-1}\left(\frac{Q \frac{\omega_0^2}{\omega_0^2} - \frac{\omega^2}{\omega}}{\omega_0}\right). \] (10.34)

**Helpful Tip**

We see immediately that resonances are completely determined by two parameters: the resonance frequency and the quality factor (or equivalently the bandwidth).
10.6.2 The Bandwidth

If we consider the power delivered to the circuit’s resistance, \( P = I^2 R \), we will find that\(^{[10.35]}\)

\[
I^2(\omega) = \frac{I_0^2}{1 + \left[ Q \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right]^2}
\]

has a Lorentzian line shape. We define the bandwidth of this circuit response to be the full width at half maximum (FWHM) of the power that it delivers to its load, the resistance in this case. At \( \omega = \omega_0 \), the denominator of Equation (10.35) is 1; the power will be half as big when the denominator is 2. To make this development more convenient, we define \( w = \omega/\omega_0 \) so that

\[
2 = 1 + \left[ Q \left( w - \frac{1}{w} \right) \right]^2
\]

at the band edges. Solving for \( w \), we find that

\[
1 = \left[ Q \left( w - \frac{1}{w} \right) \right]^2
\]

\[
\pm 1 = Q \left( w - \frac{1}{w} \right)
\]

\[
0 = Qw^2 \pm w - Q
\]

\[
w = \pm \frac{1 \pm \sqrt{1 + 4Q^2}}{2Q}.
\]

The radical is bigger than 1 so, to get positive frequencies, we must choose the positive radical. Then the bandwidth will be the range of frequencies between these two band edges, or

\[
\frac{\Delta f}{f_0} = \frac{\Delta \omega}{\omega_0} = \Delta w = \frac{1 + \sqrt{1 + 4Q^2}}{2Q} - \frac{-1 + \sqrt{1 + 4Q^2}}{2Q} = \frac{1}{Q} \text{ or } Q = \frac{f_0}{\Delta f}.
\]

Once again we see that larger values of \( Q \) (smaller resistances) result in a narrower passband, \( \Delta f \).

**Checkpoint**

What is meant by the response curve of a resonance in this experiment? All inductors have some resistance. Does this limit our ability to design a \( RLC \) circuit to respond only to an arbitrarily narrow frequency range of input signal? Why or why not?

If we recall correctly from Experiment 7, we should have noticed that suddenly changing the applied voltage (using a square wave in that case) caused the circuit to oscillate until circuit losses absorbed the energy in the oscillations. This circuit response has very important
consequences for the communications industry. We observed that larger circuit resistance (or other losses) caused those ‘transient responses’ to decay more quickly. Having a circuit that can respond quickly allows the circuit to process information more quickly as well. However, today we learned that the bandwidth also depends upon losses

\[
\Delta \omega = \frac{\omega_0}{Q} = \omega_0 \frac{R}{\omega_0 L} = \frac{R}{L}
\]

so to process more information, we must also increase the width of the circuit’s passband. Although we have invented several other kinds of filter circuits to separate a wanted channel from hundreds of unwanted channels, we have not been able to get around this fundamental fact. Evidently a higher information rate simultaneously always requires a wider passband.

### 10.6.3 Logarithmic Unit Scales

Many times in science we discuss quantities on a logarithmic scale. Logarithmic scales must always be defined relative to a chosen reference, \( M = 10 \text{ dB log}_{10} \left( \frac{m}{m_0} \right) \). The units of logarithmic scales are deci-Bels (dB) after Alexander Graham Bell; note that 10 deci = 1. In the case of resonance, the reference power is the power at resonance, \( \omega = \omega_0 \), so that on the dB scale the power is

\[
P_{\text{dB}} = 10 \text{ dB log}_{10} \left( \frac{P(\omega)}{P(\omega_0)} \right) = 10 \text{ dB log}_{10} \left( \frac{I^2(\omega)}{I_0^2} \right) = 20 \text{ dB log}_{10} \left( \frac{I(\omega)}{I_0} \right) \quad (10.37)
\]

At the band edges, \( \frac{I^2}{I_0^2} = \frac{1}{2} \) and \( 10 \text{ dB log}_{10}(\frac{1}{2}) = -3 \text{ dB} \), so we frequently refer to the “-3dB points” to represent the band edges. Also at the band edges

\[
\frac{I^2(\omega)}{I_0^2} = \frac{1}{2} \quad \text{so} \quad \frac{I(\omega)}{I_0} = \frac{1}{\sqrt{2}} \approx 0.707. \quad (10.38)
\]

When the power is down to half, the current is only down to 70.7%. We can use this fact to get an estimate of the bandwidth very quickly.

**Checkpoint**

What is a “-3dB point”?

### 10.6.4 Phase at the Band Edges

We also wish to discuss how the phase shift predicted by Equation (10.32) varies across the passband. It is easiest to see that at resonance the phase shift is \( \varphi = 0 \). At the lower band
edge,

\[
\omega_- \approx \omega_0 - \frac{\Delta \omega}{2} = \omega_0 - \frac{\omega_0}{2Q} = \frac{2Q - 1}{2Q} \omega_0
\]

\[
Q \frac{\omega_-^2 - \omega_0^2}{\omega_0 \omega_-} = Q \left( \frac{\omega_0}{\omega_-} - \frac{\omega_-}{\omega_0} \right) = Q \left( \frac{2Q - 1}{2Q - 1} - \frac{2Q - 1}{2Q} \right) = \frac{4Q - 1}{4Q - 2} \approx 1
\]

\[
\varphi_- \approx \tan^{-1} 1 = \frac{\pi}{4} = 45^\circ.
\]

In this case the argument of the response’s sine function reaches each value (2\pi for example) before the argument of the stimulus’ sine function. Similarly, \( \omega_+ \approx \omega_0 + \frac{\Delta \omega}{2} \),

\[
Q \frac{\omega_+^2 - \omega_0^2}{\omega_0 \omega_+} = -\frac{4Q + 1}{4Q + 2} \approx -1,
\]

and \( \varphi_+ \approx -\frac{\pi}{4} = -45^\circ \). In this case the argument of the response’s sine function is always \( \pi/4 \) less than the argument of the stimulus’ sine function. Time must therefore become larger by 1/8 period to make up this difference before the current will reach the same phase as the stimulus’ phase has now.