## Contents

**Foreword**  
5  

1. **Introduction to the Laboratory**  
   1.1 Objectives of Introductory Physics Laboratories  
   1.2 Calculus vs Non-Calculus Based Physics  
   1.3 What to Bring to the Laboratory  
   1.4 Lab Reports  
    7  

2. **Understanding Errors and Uncertainties in the Physics Laboratory**  
   2.1 Introduction  
   2.2 Some References  
   2.3 The Nature of Error and Uncertainty  
   2.4 Notation of Uncertainties  
   2.5 Estimating Uncertainties  
   2.6 Quantifying Uncertainties  
   2.7 How to Plot Data in the Lab  
   2.8 Fitting Data (Optional)  
   2.9 Strategy for Testing a Model  
    11  

3. **Experiment 1:**  
   **Electrostatic Forces**  
   3.1 Introduction  
   3.2 The Experiment  
   3.3 Composing and Presenting the Data  
   3.4 Analysis  
   3.5 Conclusions  
    25  

4. **Experiment 2:**  
   **Equipotentials and Electric Fields**  
   4.1 Introduction  
   4.2 Theory  
   4.3 Finding electric field magnitude  
   4.4 Analysis  
   4.5 Conclusions  
    41  

5. **Experiment 3:**  
   **Ohm’s ‘Law’**  
    55
5.1 Introduction .................................................. 55
5.2 Analysis ....................................................... 65
5.3 Conclusions ................................................... 65

6. **Experiment 4:**
   Electric Currents and Circuits .............................. 67
   6.1 Introduction ................................................ 67
   6.2 Apparatus ................................................... 72
   6.3 Analysis ..................................................... 77
   6.4 Conclusions ................................................ 78

7. **Experiment 5:**
   RC Circuits .................................................... 79
   7.1 Introduction ................................................ 79
   7.2 Apparatus ................................................... 87
   7.3 Procedure .................................................. 88
   7.4 Analysis ..................................................... 93
   7.5 Conclusions ................................................ 94
   7.6 Appendix: The exponential decay law .................... 96

8. **Experiment 6:**
   Magnetic Force on a Current Carrying Wire ................ 99
   8.1 Introduction ................................................ 99
   8.2 Magnetic Field in Magnet Gap ............................ 101
   8.3 Apparatus and Procedure ................................ 101
   8.4 Analysis ..................................................... 111
   8.5 Conclusions ................................................ 111
   8.6 Appendix: The Uncertainty in Profile Integral ........ 112

9. **Experiment 7:**
   Electromagnetic Oscillations ................................ 113
   9.1 Introduction ................................................ 113
   9.2 Damped Oscillations in RLC Circuits ..................... 118
   9.3 Experimental set-up ...................................... 121
   9.4 Procedure .................................................. 122
   9.5 Analysis ..................................................... 124
   9.6 Conclusions ................................................ 125

10. **Experiment 8:**
    Electromagnetic Resonance ................................ 127
    10.1 Introduction ............................................... 127
    10.2 The Experiment .......................................... 131
    10.3 Procedure ................................................ 132
    10.4 Analysis ................................................... 136
Welcome to the general physics laboratory! This laboratory experience is designed to guide your learning of fundamental concepts of experimentation and data collection, delivered through the medium of hands-on experiments in electricity and magnetism. As a student, you should be aware that you and your colleagues will have a broad set of backgrounds in math, science, and writing and a similarly broad set of career trajectories. Even with the diversity of participants in a course such as this, everyone can share an appreciation of the scientific process. It is our job as instructors (TAs, faculty, and other assistants) to help facilitate this learning independent of your preparation level. Some will find this easier than others, but we will have done our job if you, regardless of background, walk away appreciating a little more deeply what it means for a scientist to claim that “I know something” based on experiments.

Some passages of text have been emphasized and color coded to make finding them later more convenient.

**Checkpoint**

Checkpoints are intended to cause the student to be sure he is understanding and remembering the material before continuing to waste his time.

**Helpful Tip**

Helpful tips offer the student an opportunity to learn a shortcut or otherwise to make better use of his effort.

**Historical Aside**

A Historical Aside informs the student of some of the history associated with the discussion topic. History itself is not helpful in performing the experiment or in understanding the physics; however, it sometimes helps the student understand why we do things as we do.
WARNING

Warnings are exactly what they seem. Defying warnings can result in some personal injury (likely not serious), in some disruption of the apparatus (time-consuming to repair), etc.

General Information

General Information is usually very helpful with respect to understanding the discussion topic in the broader context of the physical world.

We hope these decorations improve the student’s experience and help him/her to learn to experiment more effectively.
Chapter 1

Introduction to the Laboratory

Physics is an experimental science. As part of basic education in Physics, students learn both physical principles and problem solving (130/135 lecture) and concepts of experimental practice and analysis (136). Physics 136-2 is designed to provide an introduction to experimental techniques in the laboratory, focused on experiments in electricity and magnetism. We will build on the concepts covered in mechanics and use them to explain the indirect observations of electric charges and currents. Since electricity itself is invisible, these studies are considerably more abstract than students are accustomed; however, the process of using a set of tools to yield data and then of analyzing the data to reach conclusions is the same.

This lab is independent from the lecture course and covers independent concepts. The topics of the lecture serve as examples that we will explore in the lab to learn why we trust and how to believe in the physical principles studied in lecture. The schedule of topics in each lecture may not correspond directly with the material in the lab, which will be focused on measuring physical phenomena. These two components complement each other, but they seldom track each other. Taken together, 130/135 and 136 should provide the knowledge, problem solving skills, intuition, and practical experience with apparatus and data collection expected of a first year student in college-level physics.

1.1 Objectives of Introductory Physics Laboratories

In this course, students should expect to advance several learning goals that are broadly relevant in science, technology, and general understanding of human knowledge. These objectives are outlined by the American Association of Physics Teachers at http://www.aapt.org/Resources/policy/goaloflabs.cfm:

- Develop experimental and analytical skills for both theoretical problems and data.
- Appreciate the “Art of Experimentation” and what is involved in designing and analyzing a data-driven investigation, including inductive and deductive reasoning.
- Reinforce the concepts of physics through conceptual and experiential learning.
- Understand the role of direct observation as the basis for knowledge in physics.
- Appreciate scientific inquiry into creatively exploring how the world works.
• Facilitate communication skills through informative, succinct written reports.
• Develop collaborative learning skills through cooperative work.

1.2 Calculus vs Non-Calculus Based Physics

The same set of experiments are given to students in both calculus-based and algebra-based physics courses. The work in this laboratory is designed to be independent of calculus, but it is natural that students with more math background can better appreciate the subtleties of the physics probed in these experiments. Calculus is never required in this course, and your grading will not be affected by your knowledge of calculus (or lack thereof).

For completeness the physical laws and principles will be presented in their most general form and that typically does require calculus; however, the student will receive the same grade if he simply ignores these derivations and goes directly to the solutions. These solutions frequently contain algebra and trigonometry but they can always be understood without resorting to calculus.

1.3 What to Bring to the Laboratory

You should bring the following items to each lab session, including the first session of the course. There is no additional textbook.

1) A bound quadrille ruled lab notebook. You must have your own, and you cannot share with your lab partner. A suitable version is sold by the Society of Physics Students in Dearborn B6. This lab notebook can be reused for future physics labs or salvaged by SPS. A scientist’s notebook is his most reliable long-term memory, his evidence that he performed the work and when, and his instructions for how to reproduce the work.

2) This Physics Laboratory 2nd Quarter lab manual. A printed copy of each relevant experiment must be brought to class each week; but the student may choose to print it himself or to purchase the printed copy from the Norris bookstore. The cost of ink and paper is commensurate with the manual’s purchase price.

3) A scientific calculator.

4) An ink pen.

5) You will need to transfer electronic data files and figures from lab to your lab reports. The lab’s computers are designed to use ‘box’ for this purpose; however this can be done by email, another cloud storage account, or a USB drive.

6) Periodically, hard documents need digitized. Each lab has a document scanner that also can utilize the students’ box accounts, but students with phone cameras might prefer to use those.
1.4 Lab Reports

**WARNING**

Our teaching assistants require signed data to accompany each report.

You will write lab reports and submit them electronically. The purpose of this exercise is both to demonstrate your work in the lab and to guide you to think a bit more deeply about what you are doing. The act of technical writing also helps improve your communication skills, which are broadly relevant far beyond the physics lab.

Appendix E of this lab manual provides some guidance on how best to prepare these reports. You should keep in mind that these are not publishable manuscripts, but concise and clear descriptions of your experiments. The electronic format allows the students to begin utilizing word processors for technical writing. Students should walk out of the lab with Data and Analysis sections mostly complete and several ideas and details to incorporate into their Purpose, Procedure, and Conclusions. One additional hour should flesh out these skeletons into report submissions. An example report is provided among the course’s CANVAS files.

In addition to background material, details of apparatus function, and instructions for gathering data, each chapter of this lab manual suggests ideas you should consider while assembling your reports. Be certain to read each chapter carefully up to the Procedures before class. You will be tasked with taking an online quiz about this material before class and prior knowledge will help you perform efficiently and correctly while in the laboratory. It is also a good idea to scan the Procedures and to examine the Analysis and Conclusions for the kinds of physics good data will demonstrate.
Chapter 2

Understanding Errors and Uncertainties in the Physics Laboratory

2.1 Introduction

We begin with a review of general properties of measurements and how measurements affect what we, as scientists, choose to believe and to teach our students. Later we narrow our scope and dwell on particular strategies for estimating what we know, how well we know it, and what else we might learn from it. We learn to use statistics to distinguish which ideas are consistent with our observations and our data.

2.1.1 Measurements, Observations, and Progress in Physics

Physics, like all natural sciences, is a discipline driven by observation. The concepts and methodologies that you learn about in your lectures are not taught because they were first envisioned by famous people, but because they have been observed always to describe the world. For these claims to withstand the test of time (and repeated testing in future scientific work), we must have some idea of how well theory agrees with experiment, or how well measurements agree with each other. Models and theories can be invalidated by conflicting data; making the decision of whether or not to do so requires understanding how strongly data and theory agree or disagree. Measurement, observation, and data analysis are key components of physics, equal with theory and conceptualization.

Despite this intimate relationship, the skills and tools for quantifying the quality of observations are distinct from those used in studying the theoretical concepts. This brief introduction to errors and uncertainty represents a summary of key introductory ideas for understanding the quality of measurement. Of course, a deeper study of statistics would enable a more quantitative background, but the outline here represents what everyone who has studied physics at the introductory level should know.

Based on this overview of uncertainty, you will perhaps better appreciate how we have come to trust scientific measurement and analysis above other forms of knowledge acquisition, precisely because we can quantify what we know and how well we know it.
2.2 Some References

The study of errors and uncertainties is part of the academic field of statistics. The discussion here is only an introduction to the full subject. Some classic references on the subject of error analysis in physics are:


2.3 The Nature of Error and Uncertainty

_Error_ is the difference between an observation and the true value.

\[
\text{Error} = \text{observed value} - \text{true value}
\]

The “observation” can be a direct measurement or it can be the result of a calculation that uses measurements; the “true” value might also be a calculated result. Even if we do not know the true value, its existence defines our “error”; but in this case we will also be unable to determine our error’s numeric value. The goal of many experiments, in fact, is to estimate the true value of a physical constant using experimental methods. When we do this our error cannot be known, so we study our apparatus and estimate our error(s) using knowledge of our measurement uncertainties.

**Example:** Someone asks you, what is the temperature? You look at the thermometer and see that it is 71°F. But, perhaps, the thermometer is mis-calibrated and the actual temperature is 72°F. There is an error of −1°F, but you do not know this. What you can figure out is the reliability of measuring using your thermometer, giving you the uncertainty of your observation. Perhaps this is not too important for casual conversation about the temperature, but knowing this uncertainty would make all the difference in deciding if you need to install a more accurate thermometer for tracking the weather at an airport or for repeating a chemical reaction exactly during large-scale manufacturing.

**Example:** Suppose you are measuring the distance between two points using a meter stick but you notice that the ‘zero’ end of the meter stick is very worn. In this case you can greatly reduce your likely error by sliding the meter stick down so that the ‘10 cm’ mark is aligned with the first point. This is a perfectly valid strategy; however, you must now subtract 10 cm from the remaining point(s) location(s). A similar strategy applies (in reverse) if your ruler’s zero is not located at the end and you must measure into a corner; in this case you must _add_ the extra length to your measurement(s).
Another common goal of experiments is to try to verify an equation. To do this we alter the apparatus so that the parameters in the equation are different for each “trial”. As an example we might change the mass hanging from a string. If the equation is valid, then the apparatus responds to these variations in the same way that the equation predicts. We then use graphical and/or numerical analysis to check whether the responses from the apparatus (measurements) are consistent with the equation’s predictions. To answer this question we must address the uncertainties in how well we can physically set each variable in our apparatus, how well our apparatus represents the equation, how well our apparatus is isolated from external (i.e. not in our equation) environmental influences, and how well we can measure our apparatus’ responses. Once again we would prefer to utilize the errors in these parameters, influences, and measurements but the true values of these errors cannot be known; they can only be estimated by measuring them and we must accept the uncertainties in these measurements as preliminary estimates for the errors.

2.3.1 Sources of Error

No real physical measurement is exactly the same every time it is performed. The uncertainty tells us how closely a second measurement is expected to agree with the first. Errors can arise in several ways, and the uncertainty should help us quantify these errors. In a way the uncertainty provides a convenient ‘yardstick’ we may use to estimate the error.

- **Systematic error**: Reproducible deviation of an observation that biases the results, arising from procedures, instruments, or ignorance. Each systematic error biases every measurement in the same direction, but these directions and amounts vary with different systematic errors.

- **Random error**: Uncontrollable differences from one trial to another due to environment, equipment, or other issues that reduce the repeatability of an observation. They may not actually be random, but deterministic (if you had perfect information): dust, electrical surge, temperature fluctuations, etc. In an ideal experiment, random errors are minimized for precise results. Random errors are sometimes positive and sometimes negative; they are sometimes large but are more often small. In a sufficiently large sample of the measurement population, random errors will average out.

Random errors can be estimated from statistical repetition and systematic errors can be estimated from understanding the techniques and instrumentation used in an observation; many systematic errors are identified while investigating disagreement between different experiments.

Other contributors to uncertainty are not classified as ‘experimental error’ in the same scientific sense, but still represent difference between measured and ‘true’ values. The challenges of estimating these uncertainties are somewhat different.

- **Mistake, or ‘illegitimate errors’**: This is an error introduced when an experimenter does something wrong (measures at the wrong time, notes the wrong value).
These should be prevented, identified, and corrected, if possible, and ideally they should be completely eliminated. Lab notebooks can help track down mistakes or find procedures causing mistakes.

- **Fluctuations:** Sometimes, the variability in a measurement from its average is not a random error in the same sense as above, but a physical process. Fluctuations can contain information about underlying processes such as thermal dynamics. In quantum mechanics these fluctuations can be real and fundamental. They can be treated using similar statistical methods as random error, but there is not always the desire or the capacity to minimize them. When a quantity fluctuates due to underlying physical processes, perhaps it is best to redefine the quantity that you want to measure. (For example, suppose you tried to measure the energy of a single molecule in air. Due to collisions this number fluctuates all over the place, even if you could identify a means to measure it. So, we invent a new concept, the *temperature*, which is related to the average energy of molecules in a gas. Temperature is something that we can measure, and assign meaningful uncertainties to. Because of physical fluctuations caused by molecular collisions, temperature is a more effective measurement than one molecule’s energy. Temperature reflects the aggregate average of all of the molecules and, as such, fluctuates far less.)
2.3.2 Accuracy vs. Precision

Errors and uncertainties have two independent aspects:

- **Accuracy**: Accuracy is how closely a measurement comes to the ‘true’ value. It describes how well we eliminate systematic error and mistakes.

- **Precision**: Precision is how exactly a result is determined without referring to the ‘true’ value. It describes how well we suppress random errors and thus how well a sequence of measurements of the same physical quantity agree with each other.

It is possible to acquire two precise, but inaccurate, measurements using different instruments that do not agree with each other at all. Or, you can have two accurate, but imprecise, measurements that are very different numerically from each other, but statistically cannot be distinguished.

2.4 Notation of Uncertainties

There are several ways to write various numbers and uncertainties, but we will describe our data using **Absolute Uncertainty**: The magnitude of the uncertainty of a number in the same units as the result. We use the symbol \( \delta x \) for the uncertainty in \( x \), and express the result as \( x \pm \delta x \).

*Example*: For an uncertainty \( \delta x = 6 \text{ cm} \) in a length measurement \( L \) of \( x = 2 \text{ meters} \), we would write \( L = (2.00 \pm 0.06) \text{ m} \). Note that \( x \) and \( \delta x \) have the same number of digits after the decimal point. In fact, \( \delta x \) tells us how many digits in \( x \) are truly measurable and allows us to discard the noise; because of this \( x \) and \( \delta x \) always have the same number of decimal places.

2.5 Estimating Uncertainties

The process of estimating uncertainties requires practice and feedback. Uncertainties are always due to the measuring tool and to our proficiency with using it.

2.5.1 Level of Uncertainty

How do you actually estimate an uncertainty? First, you must settle on what the quantity \( \delta x \) actually means. If a value is given as \( x \pm \delta x \), what does the range \( \pm \delta x \) mean? This is called the level of confidence of the results.

Assuming no systematic biases, \( x - \delta x < \text{true value} < x + \delta x \) 68% of the time. There are valid reasons to specify tolerances much greater than the statistical uncertainty. For example,
manufacturers cannot afford to have 32% of their products returned. But scientists generally use 68% confidence levels.

**Helpful Tip**

Frequently, students list “human error” among the reasons why predictions disagree with measurements. Actually, it is the tools we use that have limitations. The “human error” in reading e.g. a meter stick should be included in the measurement tolerance and compounded with other measurement uncertainties. In this case, the sigma for the comparison contains these “human errors” and cannot be the reason the difference is greater than the sigma. Humans can design micrometers and interferometers to measure better than meter sticks. Our tools are limited, but humans are more versatile.

### 2.5.2 Reading Instrumentation

Measurement accuracy is limited by the tools used to measure. In a car, for example, the speed divisions on a speedometer may be only every 5 mph, or the digital readout of the odometer may only read down to tenths of a mile. To estimate instrumentation accuracy, assume that the uncertainty is one half of the smallest division that can be unambiguously read from the device. **Instrumentation accuracy must be recorded during laboratory measurements.** In many cases, instrument manufacturers publish specification sheets that detail their instrument’s errors more thoroughly. In the absence of malfunction, these specifications are reliable; however, ‘one half of the smallest division’ might not be very reliable if the instrument has not been calibrated recently.

### 2.5.3 Experimental precision

Even on perfect instruments, if you measure the same quantity several times, you will obtain several different results. For example, if you measure the length of your bed with a ruler several times, you typically find a slightly different number each time. The bed and/or the ruler could have expanded or contracted due to a change in temperature or a slightly different amount of tension. Your eye might not be properly aligned with the ruler and the...
bed so that parallax varies the measurements. These unavoidable uncertainties are always present to some degree in an observation. In fact, if you get the same answer every time, you probably need to estimate another decimal place (or even two). Even if you understand their origin, the randomness cannot always be controlled. We can use statistical methods to quantify and to understand these random uncertainties. Our goal in measuring is not to get the same number every time, but rather to acquire the most accurate and precise measurements that we can.

### 2.6 Quantifying Uncertainties

Here we note some mathematical considerations of dealing with random data. These results follow from the central limit theorem in statistics and analysis of the normal distribution. This analysis is beyond the scope of this course; however, the distillation of these studies are the point of this discussion.

#### 2.6.1 Mean, Standard Deviation, and Standard Error

Statistics are most applicable to very large numbers of samples; however, even 5-10 samples benefit from statistical treatment. Statistics greatly reduce the effect of truly random fluctuations and this is why we utilize this science to assist us in understanding our observations. But it is still imperative that we closely monitor our apparatus, the environment, and our precision for signs of systematic biases that shift the mean.

With only a few samples it is not uncommon \( \left( \frac{1}{2N} \right) \) for all samples to be above (or below) the distribution’s mean. In these cases the averages that we compute are biased and not the best representation of the distribution. These samples are also closely grouped so that the standard error is misleadingly small. This is why it is important to develop the art of estimating measurement uncertainties in raw data as a sanity check for such eventualities.

**The mean**

Suppose we collect a set of measurements of the *same* quantity \( x \), and we label them by an integer index \( i \): \( \{x_i\} = (x_1, x_2, \ldots, x_N) \). What value do we report from this set of identical measurements? We want the mean, \( \mu \), of the population from which such a data set was randomly drawn. We can approximate \( \mu \) with the *sample mean* or average of this particular set of \( N \) data points:

\[
\mu \approx \bar{x} = \frac{1}{N} \sum_{i} x_i
\]  

(2.1)

Of course, this is not the true mean of the population, because we only measured a small subset of the total population. But it is our best guess and, statistically, it is an *unbiased predictor* of the true mean \( \mu \).
The standard deviation

How precisely do we know the value of \( x \)? To answer this question of statistical uncertainty based on the data set \( \{x_i\} \), we consider the squared deviations from the sample mean \( \bar{x} \). The sample variance \( s^2_{x} \) is the sum of the squared deviations divided by the ‘degrees of freedom’ (DOF). For \( N \) measurements the DOF for variance is \( N - 1 \). (The origin of the \( N - 1 \) is a subtle point in statistics. Ask if you are interested.) The sample standard deviation, \( s_{x} \), is the square root of the sample variance of the measurements of \( x \).

\[
s_{x} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{\text{DOF}}} \tag{2.2}
\]

The sample standard deviation is our best ‘unbiased estimate’ of the true statistical standard deviation \( \sigma_{x} \) of the population from which the measurements were randomly drawn; thus it is what we use for a 68% confidence interval for one measurement (i.e. each of the \( x_i \)).

The standard error

If we do not care about the standard deviation of one measurement but, rather, how well we can rely on a calculated average value, \( \bar{x} \), then we should use the standard error or standard deviation of the mean \( s_{\bar{x}} \). This is found by dividing the sample standard deviation by \( \sqrt{N} \):

\[
s_{\bar{x}} = \frac{s_{x}}{\sqrt{N}} \tag{2.3}
\]

If we draw two sets of random samples from the same distribution and compute the two means, the two standard deviations, and the two standard errors, then the two means will agree with each other within their standard errors 68% of the time.

2.6.2 Reporting Data

Under normal circumstances, the best estimate of a measured value \( x \) predicted from a set of measurements \( \{x_i\} \) is given by \( x = \bar{x} \pm s_{\bar{x}} \). Statistics depend intimately upon large numbers of samples and none of our experiments will obtain so many samples. Our uncertainties will have 1-2 significant figures of relevance and the uncertainties tell us how well we know our measurements. Therefore, we will round our uncertainties, \( \delta m \), to 1-2 significant figures and then we will round our measurements, \( m \), to the same number of decimal places; \((3.21 \pm 0.12) \text{ cm}, (434.2 \pm 1.6) \text{ nm}, \text{ etc.} \)

2.6.3 Error Propagation

One of the more important rules to remember is that the measurements we make have a range of uncertainty so that any calculations using those measurements also must have a
commensurate range of uncertainty. After all the result of the calculation will be different for each number we should choose even if it is within range of our measurement.

We need to learn how to propagate uncertainty through a calculation that depends on several uncertain quantities. Final results of a calculation clearly depend on these uncertainties, and it is here where we begin to understand how. Suppose that you have two quantities $x$ and $y$, each with an uncertainty $\delta x$ and $\delta y$, respectively. What is the uncertainty of the quantity $x + y$ or $xy$? Practically, this is very common in analyzing experiments and statistical analysis provides the answers disclosed below.

For this course we will operate with a set of rules for uncertainty propagation. It is best not to round off uncertainties until the final result to prevent accumulation of rounding errors. Let $x$ and $y$ be measurements with uncertainty $\delta x$ and $\delta y$ and let $c$ be a number with negligible uncertainty. We assume that the errors in $x$ and $y$ are uncorrelated; when one value has an error, it is no more likely that the other value’s error has any particular value or trend. We use our measurements as described below to calculate $z$ and the propagated uncertainty in this result ($\delta z$).

- **Multiplication by an exact number:** If $z = cx$, then
  \[ \delta z = c \delta x \]  
  \[ (2.4) \]

- **Addition or subtraction by an exact number:** If $z = c + x$, then
  \[ \delta z = \delta x \]  
  \[ (2.5) \]

- **Addition or subtraction:** If $z = x \pm y$, then
  \[ \delta z = \sqrt{(\delta x)^2 + (\delta y)^2} \]  
  \[ (2.6) \]

- **Multiplication or division:** If $z = xy$ or $z = \frac{x}{y}$, then
  \[ \frac{\delta z}{z} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2} \]  
  \[ (2.7) \]

- **Power:** If $z = x^c$, then
  \[ \frac{\delta z}{z} = c \frac{\delta x}{x} \]  
  \[ (2.8) \]

The important pattern in these rules is that when you combine multiple uncertainties, you do not add them directly, but rather you square them, add, and then take the square root. The reason for this is intuitive: if one error is randomly positive, the other one is sometimes negative, which reduces the total error. Therefore, it is incorrect to estimate the combination of two uncertainties as their sum since this overestimates the average size of the combined error.
2.6.4 (Essential) Significant Figures

**WARNING**

Failure to adhere to the following protocol *will* result in point deductions.

The *significant figures* of a number are the digits in its representation that contribute to the precision of the number. In practice, we assume that all digits used to write a number are significant (except leading zeroes). Therefore, completely uncertain digits should not be used in writing a number and results should be rounded to the appropriate significant figure. The noise in our measurements should be discarded. For example, you should not express your height as 70.056 inches if your uncertainty is $\pm 0.1$ inch. It would more appropriately be written as 70.1 inches. Uncertainties specified using only significant digits are always $\pm 5$ times a power of 10; the least significant displayed digit was the result of rounding up or down by as much as 0.5 of that digit. Usually we know our uncertainty to be something close to this but yet different. Further, results of simple calculations should not increase the number of significant digits. Calculations transform our knowledge; they do not increase our knowledge. The rounding should be performed at the final step of a calculation to prevent rounding errors at intermediate steps from propagating through your work but one or two extra digits suffice to prevent this.

Zeros are also considered significant figures. If you write a number as 1,200, we assume there are four significant digits. If you only mean to have two or three, then it is best to use scientific notation: $1.2 \times 10^3$ or $1.20 \times 10^3$. Leading zeros are not considered significant: 0.55 and 0.023 have just two significant figures. After some time the decimal point frequently gets obscured, but the ‘0’ and the space allows us to realize that this is 0.55 *not* ‘55.’

There are some guidelines for tracking significant figures throughout mathematical manipulation. This is useful as a general method to keep track of the precision of a number so as not to carry around extra digits of information, but you should generally be using more formal error estimates from Sections 2.5 and 2.6 for reporting numbers and calculations in the physics lab.

- **Addition and Subtraction:** The result is known to the decimal place of the *least* precise input number.

  *Example:* $45.37 + 10 = 55$, not 55.37 or 55.4

  *Why?* $\delta = \sqrt{0.005^2 + 0.5^2} = 0.5$

  Where we used the sum formula Equation (2.6).

- **Multiplication and Division:** The result is known to as many significant figures as are in the least precise input number.

  *Example:* $45.4 \times 0.25 = 11$, not 11.4

  *Why?* $\delta = 11 \sqrt{\left(\frac{0.05}{45}\right)^2 + \left(\frac{0.005}{0.25}\right)^2} = 0.2 > 0.05$
CHAPTER 2: UNCERTAINTIES

Where we used the product formula Equation (2.7).

Example: If you measure a value on a two-digit digital meter to be 1.0 and another value to be 3.0, it is incorrect to say that the ratio of these measurements is 0.3333333, even if that is what your calculator screen shows you. The two values are measurements; they are not exact numbers with infinite precision. Since they each have two significant digits, the correct number to write down is 0.33. If this is an intermediate result, then 0.333 or 0.3333 are preferred, but the final result must have two significant digits.

For this lab, you should use proper significant figures for all reported numbers including those in your notebook. We will generally follow a rule for significant figures in reported numbers: calculate your uncertainty to two significant figures, if possible, using the approach in Sections 2.5 and 2.6, and then use the same level of precision in the reported error and measurement. This is a rough guideline, and there are times when it is more appropriate to report more or fewer digits in the uncertainty. However, it is always true that the result must be rounded to the same decimal place as the uncertainty. The uncertainty tells us how well we know our measurement.

2.7 How to Plot Data in the Lab

Plotting data correctly in physics lab is somewhat more involved than just drawing points on graph paper. First, you must choose appropriate axes and scales. The axes must be scaled so that the data points are spread out from one side of the page to the other. Axes must always be labeled with physical quantity plotted and the data’s units. Then, plot your data points on the graph. Ordinarily, you must add error bars to your data points, but we forgo this requirement in the introductory labs. Often, we only draw error bars in the vertical direction, but there are cases where it is appropriate to have both horizontal and vertical error bars. In this course, we would use one standard deviation (standard error if appropriate for the data point) for the error bar. This means that 68% of the time the ‘true’ value should fall within the error bar range.

Do not connect your data points by line segments. Rather, fit your data points to a model (often a straight line), and then add the best-fit model curve to the figure. The line, representing your theoretical model, is the best fit to the data collected in the experiment. Because the error bars represent just one standard deviation, it is fairly common for a data point to fall more than an error bar away from the fit line. This is OK! Your error bars are probably too large if the line goes through all of them! Since 32% of your data points are more than 1σ away from the model curve, you can use this fact to practice choosing appropriate uncertainties in your raw data.

Some of the fitting parameters are usually important to our experiment as measured values. These measured parameters and other observations help us determine whether the fitting model agrees or disagrees with our data. If they agree, then some of the fitting parameters might yield measurements of physical constants.
CHAPTER 2: UNCERTAINTIES

2.8 Fitting Data (Optional)

Fully understanding this section is not required for Physics 136. You will use least-squares fitting in the laboratory, but we will not discuss the mathematical justifications of curve fitting data. Potential physics and science majors are encouraged to internalize this material; it will become an increasingly important topic in upper division laboratory courses and research and it will be revisited in greater detail.

In experiments one must often test whether a theory describes a set of observations. This is a statistical question, and the uncertainties in data must be taken into account to compare theory and data correctly. In addition, the process of ‘curve fitting’ might provide estimates of parameters in the model and the uncertainty in these parameter estimations. These parameters tailor the model to your particular set of data and to the apparatus that produced the data.

Curve fitting is intimately tied to error analysis through statistics, although the mathematical basis for the procedure is beyond the scope of this introductory course. This final section outlines the concepts of curve fitting and determining the ‘goodness of fit’. Understanding these concepts will provide deeper insight into experimental science and the testing of theoretical models. We will use curve fitting in the lab, but a full derivation and statistical justification for the process will not be provided in this course. The references in Section 2.2, Wikipedia, advanced lab courses, or statistics textbooks will all provide a more detailed explanation of data fitting.

2.8.1 Least-Squares and Chi-Squared Curve Fitting

Usually data follows a mathematical model and the model has adjustable parameters (slope, y-intercept, etc.) that can be optimized to make the model fit the data better. To do this we compute the vertical distance between each data point and the model curve, we add together all of these distances, and then we adjust all of the parameters to minimize this sum. This strategy is the “least squares” algorithm for fitting curves.

Scientific curve fits benefit from giving more precise data points a higher weight than points that are less well-known. This strategy is “chi squared” curve fitting. These algorithms may be researched readily on the internet.

2.9 Strategy for Testing a Model

2.9.1 A Comparison of Measurements

If we have two independent measurements, \( X_1 = x_1 \pm \delta x_1 \) and \( X_2 = x_2 \pm \delta x_2 \), of the same physical quantity and having the same physical units, then we will conclude that they agree if the smaller plus its \( \delta \) overlaps with the larger minus its \( \delta \).

22
A disagreement could mean that the data contradicts the theory being tested, but it could also just mean that one or more assumptions are not valid for the experiment; perhaps we should revisit these. Disagreement could mean that we have underestimated our errors (or even have overlooked some altogether); closer study of this possibility will be needed. Disagreement could just mean that this one time the improbable happened. These possibilities should specifically be mentioned in your Analysis according to which is most likely, but further investigation will await another publication.

**Helpful Tip**

One illegitimate source of disagreement that plagues students far too often is simple math mistakes. When your data doesn’t agree and it isn’t pretty obvious why, hide your previous work and repeat your calculations very carefully to make sure you get the same answer twice. If not, investigate where the two calculations began to differ.
Chapter 3

Experiment 1: Electrostatic Forces

3.1 Introduction

It was known since the time of the ancient Greeks that amber rubbed with fur would become “electrified” and attract small objects, an effect also easily seen by rubbing a piece of plastic with wool. The word “electricity” in fact comes from the Greek name for amber.

Historical Aside

During Benjamin Franklin’s time, such astonishing electrical phenomena had been observed that widely-attended public electrical displays had become popular. A public presentation of this kind so strongly impressed Franklin that he bought the lecturer’s equipment, and began investigating electrical phenomena on his own, in parallel with other efforts already underway in Europe.

Others had already found that there were two kinds of electrical charge, and that charges of the same kind repel while charges of opposite kind attract each other. Franklin designated the two kinds of charge as “positive” and “negative.” But while qualitative understanding was developing rapidly, a quantitative understanding of the forces between electrically charged objects was still lacking.

It was Charles Augustin Coulomb, a French scientist, who first quantitatively measured the electrical attraction and repulsion between charged objects and established that the force was proportional to the product of the charges and inversely proportional to the square of the distance between them. In mks units, the electrostatic force, $F_e$, that charges $q_1$ and $q_2$ a distance $r$ apart exert on each other is

$$F_e = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}.$$  (3.1)
The force acts in a direction along the straight line connecting the two charges \((\mathbf{r})\), and the force is repulsive when \(q_1\) and \(q_2\) are both positive or both negative, corresponding to a positive value of \(q_1 q_2\). The force is attractive when the charges have opposite sign so that \(q_1 q_2\) is negative. The quantity \(\varepsilon_0\), called the permittivity constant, is equal to

\[
\varepsilon_0 = 8.854 \times 10^{-12} \text{Coulomb}^2/(\text{Newton-meter}^2),
\]

and assures that the force will be in Newtons (N) when the charge is expressed in Coulombs (C) and the distance is in meters (m).

**Historical Aside**

The gravitational force similarly exhibits an inverse square dependence on distance between two point masses. Gravitational forces differ, however, by being always attractive, never repulsive, and by being inherently weaker, with the electrostatic repulsion between two protons being \(10^{36}\) times greater than their gravitational attraction.

This might seem puzzling. Gravitational forces involving massive objects can be literally strong enough to move the Earth, constantly acting to keep it in a nearly circular orbit around the Sun, and certainly we experience gravitational forces on ourselves very directly. But the electrostatic forces between pairs of objects in this laboratory are barely strong enough to lift small bits of lint or paper. Even considering the large mass of the earth, this might seem inconsistent with the statement about electrostatic forces being \(10^{36}\) times stronger than gravitational forces.

The weakness of electrostatic forces between different everyday objects reflects the fact that matter consists of almost exactly equal numbers of positively charged protons and negatively charged electrons thoroughly intermingled with one another, mainly in the form of atoms whose electrons move around positively charged nuclei consisting of protons and neutrons. The electron and proton have equal but opposite charge \((\pm q = e = 1.602 \times 10^{-19} \text{C})\), in mks units), and the neutron has zero charge. The forces of electrostatic attraction and repulsion acting between particles within this intimate mixture of electrons and protons are indeed substantial. But so close to perfect is the balance between the number of electrons and protons in ordinary matter, and so close to zero is the net charge, that two separate objects near each other hardly exert any electrostatic force at all. Yet if you were standing at arm’s length from someone and each of you had one percent more electrons than protons, the force of electrostatic repulsion would be sufficient to lift a weight equal to that of the entire earth.

**Checkpoint**

State Coulomb’s Law. How many types of electric charge exist?
CHAPTER 3: EXPERIMENT 1

Historical Aside

Since Ernest Rutherford discovered that atoms are composed of electrons moving around much heavier positively charged nuclei, a major question was why the tremendous electric forces between the protons and the electrons do not cause them to fuse together.

Similarly, tremendous electrical repulsive forces act between the positively charged protons confined within the atomic nucleus, and yet these forces do not usually succeed in pushing the nucleus apart. This is because of the additional strong short-range attractive nuclear forces that hold the protons and neutrons together despite the electrical repulsion.

General Information

Thus electrical forces and quantum mechanical effects acting together determine the precise properties of a material. With such enormous forces acting in balance within this intimate mixture, it is not hard to understand that matter, tending to keep its positive and negative charges in the finest balance, can have great stiffness and strength.

Also as a consequence of the electric and quantum mechanical effects that determine the properties of matter, when atoms combine to form solids it often happens that one or more electrons normally bound to each atom are able to wander around more-or-less freely in the material. Some of the electrons are not needed to form chemical bonds and can easily move from atom to atom. These are the conduction electrons in metals.

A consequence of Coulomb’s law is that if a metallic object has a net charge, the excess electrons will repel each other and be attracted to any region with net positive charge, so that the net charge tends to redistribute itself over the object’s surface. As a result, excess charge on an isolated spherical metal conductor will distribute itself uniformly on the outside of the conducting sphere. For the same reason, if a wire connected to a pipe buried in the ground (for example, to the plumbing in the building) touches the electrically charged conducting sphere, its excess charge will flow into the ground under the influence of the Coulomb force between charges. Practical applications of Coulomb’s law involve unbalanced (net) charge distributed over an extended region, such as an approximately spherical conductor in the present experiment, and not actually charge concentrated at a point.

General Information

The net charge on each object arises from individual discrete electrons and protons, but the small size of the electrons’ and protons’ charge and their large number make the distribution on each object appear smooth and continuous.
Coulomb’s law, Equation (3.1), then applies by regarding each charged object to be divided into small sub-regions, and by using Equation (3.1) to calculate the force that each such sub-region of the first object exerts on each small sub-region of the second object. We could then evaluate an appropriate vector sum to find the net force and torque of each object on the other.

Applying a mathematical procedure equivalent to that described in the previous paragraph shows that excess charge distributed uniformly over the surface of a sphere exerts a force on a small test charge a distance away as if all the charge on the sphere were concentrated at its center. For this special case Equation (3.1) ends up applying in the form given provided that the distance, \( r \), to the excess charge on the metal sphere is taken as the distance to the center of the sphere.

**Checkpoint**

In a nucleus there are several protons, all of which have positive charge. Why does the electrostatic repulsion fail to push the nucleus apart?

### 3.1.1 Conservation of Charge

Amber rubbed with fur acquires a net negative charge because some of the negatively charged electrons are pulled from the fur onto the amber, leaving the fur positively charged. Since electrifying objects by friction involves merely moving the charges from one place to another, the total charge stays the same.

**Checkpoint**

What does it mean to say that charge is conserved? An electron with a charge of \(-1.602 \times 10^{-19} \text{C}\) can combine with a positron having charge \(+1.602 \times 10^{-19} \text{C}\) to yield only uncharged products. Is charge conserved in this process?

**General Information**

This principle is far more fundamental and general however. New particles can be produced in high-energy reactions such as those at the Fermi-Lab National Accelerator. Charged particles are not merely moved from one place to another but are created. Yet in each reaction the number of newly created positively-charged particles always equals the number of new particles that are negatively charged. Since the net charge in all known physical processes stays the same, charge is said to be conserved.
3.2 The Experiment

**WARNING**

The pith balls are fragile and in short supply so do NOT touch the pith balls under any circumstance... handle only the strings.

The measurements we do will test the distance dependence in Coulomb’s law. The equipment used to produce net charge needed consists of the electrostatic generator illustrated in Figure 3.1. We also use a small pith ball hanging by a nylon line. The pith ball is a small insulating sphere with a conducting outer surface that allows excess charge to distribute itself evenly.

A known charge \( q \) is first placed on the ball, and then the generator is used to place a charge \( Q \) on the spherical generator dome. The electrostatic force acts to move the ball away from the generator dome, but is balanced by the gravitational force (and the string’s tension) tending to pull the pith ball back to its lowest position.

The measured displacement of the ball in the form of the angle between the string and a vertical line determines the component of gravitational force in the direction tending to rotate the pith ball and string back vertical; therefore, the angle of deflection measures the electrostatic force acting on the pith ball. We quickly measure the string angle for several values of the position \( x_1, x_2, \ldots, x_N \) and distance, \( R \), to the center of the dome, and immediately repeat the measurement for the first value of \( x_1 \). The \( R \) dependence of the measured electrostatic force can then be compared with that predicted by Coulomb’s law.

Figure 3.1 shows the electrostatic generator. The bottom roller is driven by a motor, causing the continuous rubber belt to turn. Friction between the belt and the wool on the bottom roller transfers charged ions between the wool and the belt so that the wool becomes positively charged and the belt becomes negatively charged. The belt carries the negative charge to the top roller where the charge is transferred to the top collector. Then, friction causes charged ions to be exchanged between the top roller and the belt and leaves the roller negatively charged and the belt positively charged.

The materials in the belt and rollers are carefully chosen to make the charge transfers have the correct sign. The region of positive charge on the belt is carried back to the bottom roller.

![Figure 3.1: A sketch of the Van de Graaff generator showing how an insulating belt transports electrons to the dome.](image-url)
A contact placed close to the bottom roller is grounded (meaning that it is attached to a nearby pipe or wire that ultimately is connected to the earth using a ground rod). The flow of charge to or from the ground cancels out the excess charge on the part of the belt near the contact. By this mechanism, the belt and roller system act as a pump, doing work on the negative electric charge against the repulsive force of the charge already on the dome, and depositing it on the dome with a high potential energy per unit charge, corresponding to a high voltage.

**Checkpoint**

How does the Van de Graaff generator operate?

The charges generated in this experiment are not dangerous, but you might experience some unpleasant, disconcerting shocks by not following the instructions precisely as given. You may also pick up charge that can subject you to a minor electrical shock, about as strong as a carpet shock, when you touch a grounded object.
The measurements to be made and later analyzed using the set-up in Figure 3.2 require three basic steps:

1) Aligning the dome with respect to the pith ball’s pivot,
2) placing a charge, $q$, on the ball and determining its value, and
3) measuring the position, $x$, of the charge $Q$ on the generator and the angle, $\theta$, between the string and a vertical line.

### 3.2.1 Aligning the Apparatus

As shown in Figure 3.4, the electrostatic generator is arranged so it can be displaced along a meter stick fastened to your laboratory bench. In addition, each set-up includes a dull aluminum sphere grounded to earth and mounted on an insulating rod of Lucite; the aluminum sphere is for use in grounding the conducting generator dome and/or the pith ball when necessary to remove the excess charge.

**Helpful Tip**

If the pith ball string moves toward or away from the protractor, the angle measurement will be distorted. In that case move the protractor by rotating the support rod on the table to get the string to move parallel to the protractor’s face.

First you need to determine the position, $x_0$, of the voltage generator at the point where the pith ball, hanging vertically, is at its center. To do this, ground the generator sphere by touching it with the dull aluminum sphere. Remove the top half of the dome, relocate one of the pith balls to the plastic screw, place the small plastic ruler across the dome’s diameter, accurately locate the center, and shift the generator’s base until the center of the pith ball is in line with the center of the protractor.

**Helpful Tip**

From the end of the lab bench, see that the motion of the large dome along the meter stick is aligned with the protractor. If necessary, ask your TA to help you optimize your apparatus.

**Figure 3.4:** The top of the dome is removable so that we can accurately match the spheres’ centers and measure the Van de Graaff’s reference position, $x_0$. 
Figure 3.3: The physics of charging pith balls. An object with negative charge is brought near the dielectric balls. Positive charge in the balls are attracted closer and negative charge is repelled farther. Since the positive charges are closer, the net force is attractive. The balls touch the rod, pick up a net negative charge, and are repelled from the rod and each other.

ball lies at the center of the sphere. (See Figure 3.4.) Note that the Van de Graaff generator is somewhat flimsy so that carelessness can cause the dome to move without the base moving commensurately. Take care to handle the generator from the base only.

If necessary, ask your teaching assistant to adjust the height of the horizontal bar supporting the pith balls so their midpoint coincides with the center of the generator sphere. Record the position of the side of the generator along the table on the clamped meter stick. This measured location along the ruler is called $x_0$ in the equations we will be using later. (See Figure 3.6.) When the base is at $x_0$, the large sphere’s center is directly under the pith ball pendulum’s pivot. Record the protractor reading for the un-deflected string as $\theta_0$. Angles toward the support rod are positive and angles toward the Van de Graaff generator are negative. Record your error and units. I would suggest a small table in your notebook (and later in your report) to hold $x_0$, $L$, $\theta_0$, $r_0$, and $m_1 \approx m_2$.

**Checkpoint**

After the Van de Graaff generator has been running and is turned off with its dome still charged, how would the charge distribution in the grounded aluminum sphere be affected by bringing it near the dome without making contact? Explain this effect in terms of the electrostatic forces acting and the properties of the metallic sphere.
Now move the generator along the ruler in the direction away from the vertical pole until the pith ball clears the dome by more than 10 cm and replace the top of the dome. If the experiment were done with the Van de Graaff generator too close to the vertical pole, the charge induced in the conducting pole would in turn exert Coulomb forces to redistribute the charge in the dome making it spherically asymmetric.

**Checkpoint**

Why is it possible to use the formula for the force between two point charges, Equation (3.1), for the force between the charged pith ball and the dome of the Van de Graaff generator when the electrified dome is not even approximately a point charge?

### 3.2.2 Charging the Pith Balls

It is wise to construct a table in your notebook to keep your $x$ vs. $\theta$ data. While you’re at it, you might consider another table to keep $\theta_0$, $x_0$, $L$, $m_1$, $m_2$, and $r_0$. I would suggest one partner stand still and read the protractor while the other places the function generator, reads $x$, and writes down the data.

**Helpful Tip**

Static charges dissipate very quickly—especially when humidity is high. Because of this it is necessary to minimize the time spent taking data.

Be sure you know exactly what to do and, once you start, continue taking measurements until you finish. Save the calculations and delayable measurements for later. Repeating the measurements only takes a little time, so there is no reason for stress if the first or second try does not go smoothly.

First, ground both pith balls by touching them with the dull aluminum sphere. Then wrap the fur around the pointed end of the rubber rod and briskly rub the rod with the fur to produce a net negative charge on the rod. Bring the charged rod close to the pith balls. A positive charge will first be induced on the sides of the pith balls closest to the rod, as shown in Figure 3.3, causing the balls to be attracted to the rod. After the rod and pith balls make contact, a negative charge (consisting of electrons) will pass from the

![Figure 3.5](image_url)

**Figure 3.5:** Illustrates the strategy for measuring the pith balls’ charges.
rod to the balls. When enough negative charge has been transferred, the balls will fly away from the rod and will repel each other. The similar balls collecting charge from the same source will acquire similar charges; we will assume these charges are identical. If the balls do not fly away from the rod within 10 seconds, they are too dry and must be moistened by breathing gently on them. The separation between the two balls once charged should be between 2-6 cm. Do not touch the pith balls or they may be partially discharged. Untangle the pith ball strings. Place the rod on the table and adjust the plastic caliper's width to be the distance between the centers of the balls; gently touching the balls with the plastic is ok. Carefully lay the calipers aside until you are done with the static charges.

### 3.2.3 Taking the Data

Lift one of the charged balls by its line and drape it over the insulated plastic peg mounted on the meter stick without touching or discharging the other ball. If you disturb the remaining pith ball’s charge before you complete your data table, you will need to begin again... Slide the electrostatic generator along the table until the charged ball is about 5-10 cm from the surface of the generator’s sphere. Use the power cord switch to run the generator for several short bursts until the ball is deflected about 5° to 10° as measured by the protractor at the line support point. You can now slide the Van de Graaff generator closer to see larger deflection angles. Set the generator to exact centimeter (cm) positions to make reading and recording the positions quicker. Record the deflection angle, \( \theta_1 \), and the horizontal position, \( x_1 \), of the generator. Now shift the generator sideways so as to decrease the deflection angle and again record \( x_2 \) and \( \theta_2 \). Make six separate measurements to obtain values of \( \theta \) for \( \Delta x = x - x_0 \) between 10 cm and 60 cm (see Figure 3.6). You will not get very good results unless you utilize at least half of the meter stick. The data obtained will be used to determine the total charge \( Q \) on the generator sphere and to verify the inverse square distance dependence in Coulomb’s law.

**WARNING**

Keep the grounded sphere far away from the pith ball and the Van de Graaff sphere. It will distort the charge distribution(s); can you explain why?

**Checkpoint**

Is humidity in the room a concern in this experiment? Why or why not? As time passes, the pith balls lose their excess charge. Where does it go?

Immediately after the last measurement, return the generator to the position of the first measurement, \( x_1 \), and record the value of \( \theta_1' \).
CHAPTER 3: EXPERIMENT 1

Helpful Tip

Now, you can slow down a little and reflect upon what you have done.

Now you can place both pith balls and the Van de Graaff dome in electrical contact with the grounded sphere to discharge everything; leave them connected so that charged air currents from adjacent experiments is continually dissipated.

How closely $\theta'_1$ agrees with $\theta_1$ provides information about how much the dissipation of the charge affected your results; one might expect that the same angle would be measured for $x_1$ both times. Don’t forget to measure the distance $r_0$ between the pith balls stored by your plastic dividers. How accurately can you estimate the positions of the pith balls? How accurately can you measure the separation between the caliper’s tips.

3.3 Composing and Presenting the Data

WARNING

Static electricity kills electronic circuitry. Keep away from the computers until everyone in the lab has discharged their apparatus.

Once you finish collecting your data, you can use Equation 3.2 to compute the charge on your two pith balls. You can measure the string’s length and the pith balls’ masses. You can note observations that might affect your data. You can even repeat the experiment once or twice to provide you with a choice or two. You can construct your report’s skeleton. There is no reason to sit idle while your classmates catch up.

You probably will have noticed by this point that the Van de Graaff generator can produce impressive electrical discharges.

Figure 3.5(b) shows the vector diagram for the three forces acting on each ball in equilibrium. They are: the tension $T$ in the nylon line, the pith balls’ weight $mg$, and the Coulomb force $F_e$. The mass $m$ is written on each ball or on the protractor where the string attaches in units of milligrams (mg) and the length of the nylon line must be measured.

1) Before you come to lab, show that if each pith ball has mass $m$, hangs from a string of length $L$, and the two are separated by a distance $r_0$ because of their charge, the charge $q$ on each is

$$q = \frac{2\pi \varepsilon_0 m g r_0^3}{\sqrt{L^2 - \left(\frac{r_0}{2}\right)^2}}. \quad (3.2)$$

2) Add this proof to your notebook.
3) Calculate the charge \( q \) (in Coulombs) on the pith balls.

4) One option in your Analysis is to discuss the number of electrons needed to make up this charge.

**Checkpoint**

Why does the experiment require using two pith balls rather than one? Prove Equation (3.2) of this lab write-up outside of class.

Consider Figure 3.6 and note that the sphere moves horizontally as \( x \) changes. \(|x - x_0|\) is the horizontal distance between the pivot and the sphere’s center. When the pith ball is deflected, the horizontal distance from the pivot is opposite to \( \theta \) and \( R_x = |x - x_0| + L \sin \theta \). The heights of the pith ball and sphere are different because the string rotated about the pivot. They were at the same height when the pith ball was \( L \) below the pivot and when the pith ball is deflected it is \( L \cos \theta \) below the pivot as shown in Figure 3.6. The difference in height between the sphere and the deflected pith ball is \( R_y = L - L \cos \theta \). The red line in Figure 3.6 shows the distance, \( R \), between the center of the generator sphere and the center of the ball and is given by

\[
R = \sqrt{R_x^2 + R_y^2} = \sqrt{(|x - x_0| + L \sin \theta)^2 + (L - L \cos \theta)^2}.
\] (3.3)

The angle \( \alpha \) between the line joining the two centers and the horizontal is

\[
\alpha = \tan^{-1}\left( \frac{R_y}{R_x} \right) = \tan^{-1}\left( \frac{L - L \cos(\theta - \theta_0)}{|x - x_0| + L \sin(\theta - \theta_0)} \right).
\] (3.4)

By resolving the forces perpendicular to the thread supporting the ball, it can be shown that the electrical force \( F_e \) acting on the ball is given in terms of the observed \( \theta \) by

\[
F_e = mg \frac{\sin(\theta - \theta_0)}{\cos(\theta - \theta_0 - \alpha)}.
\] (3.5)
CHAPTER 3: EXPERIMENT 1

If we make the assumption that \( \alpha \) is small, this force can be written as

\[
F_e = mg \tan(\theta - \theta_0)
\]  

(3.6)

Equation (3.6) can be used to calculate the force \( F_e \) for each measured value of \( \theta \) and therefore at each \( R \) in Equation (3.3). We seek to compare the observed dependence of \( F_e \) on \( R \) with that in Coulomb’s law,

\[
F_e = \frac{qQ}{4\pi\varepsilon_0 R^2}.
\]

It is possible to use the program Ga3 to have the computer to make these calculations, to plot the results, and then to fit the data to a power law equation by choosing optimum values for \( R \)’s exponential power and the constant \( A = \frac{qQ}{4\pi\varepsilon_0} \).

3.3.1 Verifying the Inverse Square Law

We will use Vernier Software’s Graphical Analysis 3.4 (Ga3) program to analyze our data. A suitable setup file for Ga3 can be downloaded from the lab’s website at

http://groups.physics.northwestern.edu/lab/electrostatic.html

First enter the raw data, \( x \) and \( \theta \), in the first two columns Position and Angle. You might want to verify that the computer is using the correct formulas to calculate the distance \( R \) between the charges

\[
\sqrt{(\text{abs}(“Position”-x0)+\text{Length}*\sin(“Angle”-Angle0))^2 + (\text{Length}*(1 - \cos(“Angle”-Angle0)))^2)/100.}
\]

Checkpoint

Why did we divide everything by 100?

Helpful Tip

Parameters \( x0 \), \( \text{Length} \), \( \text{Angle0} \), and \( \text{Mass} \) can be adjusted at the bottom left to match your measurements.

In a similar way verify the column for electrostatic force, \( F_e \), uses Equation (3.6). In this case the formula should be

\[
\text{Mass} \times 9.807 \times \tan(“Angle”-\text{Angle0})/10^6
\]
Checkpoint

Why did we divide everything by $10^6$?

Once you have the correct numbers everywhere in your table, verify that the electrostatic force is on the vertical axis and that the distance between the charges is on the horizontal axis. You should see an inverse relation on the plot where as $R$ gets larger $F_e$ becomes smaller. You would like to verify that this is an inverse square relation.

Use the mouse to drag a box around the data points so that all rows in the table turn grey. If some rows in your table do not turn grey, Data/Sort from the menu and sort in increasing $R$. Select the data again. Analyze/Curve Fit... from the menu. When the box opens, select the ‘Power’ formula from the set of ‘Stock functions’ and Try Fit. The variable $B$ should tell you to what power the data depends on $R$. Coulomb’s law predicts it to be $-2$. Be sure that the continuous model line passes among your data points.

Sometimes the fit might need a little help. In this case type in values of $A$ and $B$ to see how they fit your data. Larger $A$ makes the line move up and larger $B$ makes the line fall off quicker with distance. Once the fit is pretty good, write down the fit parameters ($A$ and $B$), select “Automatic” at top-right, and “Try Fit” again. Try to get automatic fit to work so the computer will generate uncertainties.

The computer will try to find $A$ and $B$ such that

$$F_e = A R^B$$

models your data as closely as possible. Computers are stupid; you must decide whether the model curve represents your data points. Using the value $A$ and your pith ball charge, calculate the charge on the Van de Graaff generator dome. Also indicate the sign of the charge if it is made of electrons.

Checkpoint

How many excess electrons compose the generator’s charge? Do your data indicate whether the pith ball charge and dome charge have the same sign? Do the data indicate which is $+$ and which is $-{}$?

3.4 Analysis

How did the first $\theta$ value differ when remeasured at the end, and what does this tell you about any experimental error caused by charge leaking off the pith balls and the generator? Is your exponent $B$ too large or too small? Is this consistent with charge dissipating to ground? (You might want to consider the chronological ordering of your data points while
CHAPTER 3: EXPERIMENT 1

answering this.) Does the computer’s estimated uncertainty ($\delta B$) in $B$ bracket Coulomb’s prediction of -2? See Section 2.9.1.

What are some subtle sources of error that we have not recorded? Might wind from the temperature control be significant? What about your classmates’ charges? Are the charge distributions spherically symmetric or does the presence of the other charge disturb this? If all of the $R$’s were a little larger, would this improve your agreement? What if the Van de Graaff generator tilted on its base while the experiment was in progress? What if the line between the charge centers was not very horizontal?

Is the ratio of your calculated charges approximately the same as the ratio of surface areas? Do your data satisfy Coulomb’s law to within reasonable experimental error? Explain the possible sources of any disagreement. How do the values of the charges you measured compare with your expectations? How hard would it be to place a coulomb or two of charge on the dome of the generator?

As in this case, many times we must test a theory one piece at a time when we don’t have the technology to measure all of the variables. We can determine whether $F_e \propto R^{-2}$ is consistent with our observations. We can use Coulomb’s law to calculate the pith balls’ charge. We can use Coulomb’s law and our data fit parameter $A$ to determine the dome’s charge. We can use our experience to decide whether using Coulomb’s law yields reasonable values for these charges. However, we did not measure the static charges or the dissipated charges independently of Coulomb’s law, so this “circumstantial evidence” is less than complete. If all of these pieces support Coulomb’s law, this is still pretty compelling; if some do not support Coulomb’s law it is less and less so. The difficulty in performing the experiment well makes it difficult to conclude that only Coulomb’s law’s failure can explain the discrepancies.

3.5 Conclusions

What values and units of charge did you measure? Does your data support, contradict, or say nothing about Coulomb’s law? What constant exponent did you measure? Always communicate with complete sentences and define all symbols. Your reader should be able to understand your Conclusions without reading anything else in your report. What changes might improve the experiment? Can you think of applications for anything you have used or observed?

Helpful Tip

Review Appendix E frequently while assembling your reports.
Chapter 4

Experiment 2:
Equipotentials and Electric Fields

4.1 Introduction

One way to look at the force between charges is to say that the charge alters the space around it by generating an electric field $E$. Any other charge placed in this field then experiences a Coulomb force. We thus regard the $E$ field as transmitting the Coulomb force.

To define the electric field, $E$, more precisely, consider a small positive test charge $q$ at a given location. As long as everything else stays the same, the Coulomb force exerted on the test charge $q$ is proportional to $q$. Then the force per unit charge, $F/q$, does not depend on the charge $q$, and therefore can be regarded meaningfully to be the electric field $E$ at that point.

In defining the electric field, we specify that the test charge $q$ be small because in practice the test charge $q$ can indirectly affect the field it is being used to measure. If, for example, we bring a test charge near the Van de Graaff generator dome, the Coulomb forces from the test charge redistribute the charge on the conducting dome and thereby slightly change the $E$ field that the dome produces. But secondary effects of this sort have less and less effect on the proportionality between $F$ and $q$ as we make $q$ smaller.

So many phenomena can be explained in terms of the electric field, but not nearly as well in terms of charges simply exerting forces on each other through empty space, that the electric field is regarded as having a real physical existence, rather than being a mere mathematically-defined quantity. For example when a collection of charges in one region of space move, the effect on a test charge at a distant point is not felt instantaneously, but instead is detected with a time delay that corresponds to the changed pattern of electric field values moving through space at the speed of light.

Closely associated with the concept of electric field is the pictorial representation of the field in terms of lines of force. These are imaginary geometric lines constructed so that the direction of the line, as given by the tangent to the line at each point, is always in the direction of the $E$ field at that point, or equivalently, is in the direction of the force that would act on a small positive test charge placed at that point. The electric field and the
The concept of lines of electric force can be used to map out what forces act on a charge placed in a particular region of space.

Figures 4.1(a) and 4.1(b) show a region of space around an electric dipole, with the electric field indicated by lines of force. The charges in Figure 4.1(a) are identical but opposite in sign. In Figure 4.1(b) the charges have the same sign. Above each figure is a picture of a region around charges in which grass seed has been sprinkled on a glass plate. The elongated seeds have aligned themselves with the electric field at each location, thus indicating its direction at each point.

A few simple rules govern the behavior of electric field lines. These rules can be applied to deduce some properties of the field for various geometrical distributions of charge:

1) Electric field lines are drawn such that a tangent to the line at a particular point in space gives the direction of the electrical force on a small positive test charge placed at the point.

2) The density of electric field lines indicates the strength of the \( \mathbf{E} \) field in a particular region. The field is stronger where the lines get closer together.

3) Electric field lines start on positive charges and end on negative charges. Sometimes the lines take a long route around and we can only show a portion of the line within a diagram of the kind below. If net charge in the picture is not zero, some lines will not have a charge on which to end. In that case they head out toward infinity, as shown in Figure 4.1(b).

You might think when several charges are present that the electric field lines from two charges could meet at some location, producing crossed lines of force. But imagine placing a charge where the two lines intersect. Charges are never confused about the direction of the force acting on them, so along which line would the force lie? In such a case, the electric fields add vectorially at each point, producing a single net \( \mathbf{E} \) field that lies along one specific line of force, rather than being at the intersection of two lines of force. Thus, it can be seen that none of the lines cross each other. It can also be shown that two field lines never merge to become one.
Under certain circumstances, the rules defining these field lines can be used to deduce some general properties of charges and their forces. For example, a property easily deduced from these rules is that a region of space enclosed by a spherically symmetric distribution of charge has zero electric field everywhere within that region (assuming no additional charges produce electric fields inside). Imagine first a spherically symmetric thin shell of positive charge all at a certain distance from the center. Field lines from the shell would have to be radially outward equally in all directions. If these outward pointing lines continued radially inward beneath the shell, they would have nowhere to end. Hence, the field lines must have ended at the surface charge, and there must be zero field everywhere inside. Next, suppose the spherically symmetric distribution of charge surrounding the uncharged region is not merely a thin shell. We can nevertheless consider the charge distribution to be divided up into many thin layers each at a different radius. Each layer contributes its own field lines that end at that layer, producing none of the field lines in the region enclosed by that layer. Then any point in the region of interest is inside all of the thin layers, where all the field lines have ended. We can conclude that the \( \mathbf{E} \) field is zero at any point within a region surrounded by the spherical distribution of charge.

In Figures 4.1(a) and 4.1(b) it is seen that the density of field lines is greater near the charges because the lines must converge closer together as they approach a particular charge. It can also be shown that the electric field intensity increases near conducting surfaces that are curved to protrude outward, so that they have a positive curvature. Curvature is defined as the inverse of the radius. A flat surface has zero curvature. A needle point has a very small radius and a large positive curvature. The larger the curvature of a conducting surface, the greater the field intensity is near the surface.

**Checkpoint**

What are three properties of electric lines of force? Why do electric lines of force never cross? How do the electric lines of force represent an increasing field intensity?
CHAPTER 4: EXPERIMENT 2

Checkpoint

How can you prove from properties of electric field lines that a spherical distribution of charge surrounding an uncharged region produces no electric field anywhere within the region?

Historical Aside

The lightning rod is a pointed conductor. An electrified thundercloud above it attracts charge of the opposite sign to the near end of the rod, but repels charge of the same sign to the far end. The far end is grounded, allowing its charge to move across the earth. The rod thereby becomes charged by electric induction. The cloud can similarly induce a charge of sign opposite to its own in the ground beneath it. The strongest field results near the point of the lightning rod, and is intense enough to transfer a net charge onto the airborne molecules, thus ionizing them. This produces a glow discharge, in which net electric charge is carried up on the ionized molecules in the air to neutralize part of the charge at the bottom of the cloud before it can produce a lightning bolt.

The electric field representation is not the only way to map how a charge affects the space around it. An equivalent scheme involves the notion of electric potential. The difference in
electric potential between two points A and B is defined as the work per unit charge required
to move a small positive test charge from point A to point B against the electric force. For
electrostatic forces, it can be shown that this work depends only on the locations of the
points A and B and not on the path followed between them in doing the work. Therefore,
choosing a convenient point in the region and arbitrarily assigning its electric potential to
have some convenient value specifies the electric potential at every other point in the region
as the work per unit test charge done to move a test charge between the points. It is usual
to choose either some convenient conductor or else the ground as the reference, and to assign
it a potential of zero.

**General Information**

This bears some similarity to how the gravitational potential energy was defined. We
could have considered the “electric potential energy” of a test charge in analogy with
the gravitational potential energy by considering the work, not the work per unit
charge, done in moving a test charge between two points. But just as the gravitational
potential energy itself cannot be used to characterize the gravitational field because it
depends on the test mass used, the electric potential energy similarly depends on the
test charge used.

But the force and therefore the work to move the test charge from one location to another
is proportional to its charge. Thus the work per unit charge, or electric potential difference,
is independent of the test charge used as long as the field does not vary in time, so that the
electric potential characterizes the electric field itself throughout the region of space without
regard to the magnitude of the test charge used to probe it.

It is convenient to connect points of equal potential with lines in two dimensional prob-
lems; or surfaces in the case of three dimensions. These lines are called equipotential lines;
these surfaces are called equipotential surfaces; volumes, surfaces, or lines whose points all
have the same electric potential are called equipotentials.

If a small test charge is moved so that its direction of motion is always perpendicular to
the electric field at each location, then the electric force and the direction of motion at each
point are perpendicular. No work is done against the electric force, and the potential at
each point traversed is therefore the same. Hence a path traced out by moving in a direction
perpendicular to the electric field at each point is an equipotential.

Conversely, if the test charge is moved along an equipotential, there is no change in
potential and therefore no work done on the charge by the electric field. For non-zero
electric field this can happen only if the charge is being moved perpendicular to the field at
each point on such a path. Therefore, electric field lines and equipotentials always cross at
right angles.

Figure 4.2 shows a region of space around a group of charges. The electric lines of
force are indicated with solid lines and arrows. The electric field can also be indicated by
equipotential lines, shown as dashed lines in the figure. The mapping of a region of space
with equipotential lines or, in the case of 3-D space, with equipotential surfaces, provides the same degree of information as by mapping out the electric field itself throughout the region.

**Checkpoint**

Are the electric field representation and the equipotential line representation equivalent in terms of how much information they contain about the electric field?

### 4.2 Theory

Recall that the work done by the electric force, \( \mathbf{F} \), in moving the charge from point \( a \) to point \( b \) is given by

\[
W_{ab} = \int_{a}^{b} \mathbf{F} \cdot d\mathbf{r}, \tag{4.1}
\]

where \( d\mathbf{r} \) is a small piece of the path traveled from \( a \) to \( b \). We can write this in terms of the electric field; if our charge is \( q \), then \( \mathbf{F} = q\mathbf{E} \) and

\[
W_{ab} = \int_{a}^{b} q\mathbf{E} \cdot d\mathbf{r} = q \int_{a}^{b} \mathbf{E} \cdot d\mathbf{r}. \tag{4.2}
\]

Because of this we also find it convenient to talk about the work per unit charge

\[
V_{ab} = \frac{W_{ab}}{q} = \int_{a}^{b} \mathbf{E} \cdot d\mathbf{r}. \tag{4.3}
\]

Since the electric force is conservative, we also find it convenient to introduce a potential energy function, \( U(x, y, z) \), and a potential energy per unit charge function; we call this the potential function, \( V(x, y, z) = U/q \), and define its units to be the Volt (1 V = 1 J/C). We define \( U(x, y, z) \) in such a way that total energy is conserved. Since work is a change in kinetic energy, it must correspond to an opposite change in potential energy if the total is to remain constant,

\[
W_{ab} = -\Delta U = -q\Delta V = -q \left( V(x_b, y_b, z_b) - V(x_a, y_a, z_a) \right) = q(V_a - V_b). \tag{4.4}
\]

**Checkpoint**

What are the units of potential difference? What are units of electric field?
4.2.1 Mapping equipotentials between oppositely charged conductors

The equipotential apparatus is shown in Figure 4.3. The power supply is a source of potential difference (work per unit charge) measured in Volts (V). When it is connected to the two conductors, a small amount of charge is deposited on each conductor, producing an electric field and maintaining a potential difference, identical to that of the power supply, between the two conductors. The black paper beneath the conductors is weakly conducting to allow a small current to flow. The voltage sensor measures the potential difference between the point on the paper where the probe is held and the power supply’s ground (black) lead. The voltage sensor is efficient at determining potential difference using a very small (but nonzero) current. (We will understand this better after we discuss Ohm’s law.) This small current perturbs the paper’s current slightly, but much, much less than the paper’s current itself.

Remove the electrodes left behind by the last class. Choose the conductor geometry for which you will be mapping the field. Start with a circular conductor on the terminal post furthest away from you and a horizontal bar on the terminal nearest you. Wipe away any eraser crumbs from the area of the electrodes. Mount these conductor pieces on the brass bolts which protrude from the black-coated paper. Each electrode has a raised lip around its edge on one side. This side must face down so that the raised lip makes good electrical contact with the black paper. Secure the conductors with the brass nuts. Tighten down the nuts well to ensure good electrical contact between the conductors and the paper. The banana jack away from you is red and the jack close to you is black. The positive terminal of the power supply is connected to the red banana jack and the negative terminal to the black banana jack. These jacks are connected to the bolt holding the round electrode and to the center bolt holding the bar, respectively, using wires under the apparatus.

You will use the red (positive) lead of the voltage sensor as an electric potential probe to map out $V(x, y)$ in the plane of the paper. The ‘Signal Generator’ icon at the left toggles the visibility of the power supply’s controls. You can change the disk’s potential by entering different numbers into the signal generator’s control. Before the computer will make any measurements, you must ‘Record’ on the left end of the toolbar at the bottom of the screen.

Choose a few points at random on the black paper and place the red probe lightly at these points in turn. Notice that varying the disk’s potential as described above causes the potentials of the random points in the black paper to vary commensurately. Note this observation in your Data.

4.2.2 Setting the potential difference

Adjust the power supply to maintain the desired voltage between the two conductors by following these steps. Touch the red potential probe to the round electrode and hold it there. Adjust the power supply voltage to 6.00 Volts, as read by the voltage sensor. Note that all points on the round electrode have the same voltage. The electrodes are equipotential volumes and their surfaces are equipotential surfaces. When this adjustment is completed,
remove the probe from the round electrode. Note the voltage of each of the electrodes in your Data. You are now ready to take data.

4.2.3 Mapping equipotential lines

Each equipotential line or surface is specified by the same single value of the voltage that all its points have with respect to the bar electrode. The goal is to locate points at each desired potential in order to trace out the corresponding equipotential line.

Suppose, for example, you want to find an equipotential at 5.00 Volts. Lightly place the red probe on the surface of the black paper and gently move it around until the digital voltmeter reads 5.00 Volts. This point is then at a potential of 5 Volts above that of the bar conductor. We need to determine the \((x, y)\) coordinates of this point so that we can plot it on the graph paper.

The bar is inscribed with marks at every two centimeters. One side of the bar is inscribed every two millimeters. These marks can be used as our \(x\)-coordinates. We also have a ruler that we can use to determine the \(x\), \(y\)-coordinates. Plot the point on the graph paper and draw a box, triangle, diamond, star, etc. around the point to distinguish it from dirt or stray toner. An accepted strategy is to use different shapes to represent different voltages. Now, gently drag the probe across the black paper and note that very close to this point is another point on each side of the first that also have 5.00 V potential. It would take forever to find and to plot all of the 5.00 V points because these points are arbitrarily close together. The equipotentials are continuous. Move an inch or two away from your first point, trace an arc around, find and plot another point having 5.00 Volts. Continue until you are confident that you can sketch the 5.00 V equipotential on your graph/map.

**Helpful Tip**

It is not necessary to obtain exactly 5.000 Volts on the meter. We only need to get as close to the 5.000 Volts as we can transfer to the graph paper; get within 1 mm since this is closer than we can graph anyway.

Note that the graph paper is half as big and is scaled 1:2 with respect to the apparatus.

**General Information**

In science experiments it is often important for us to notice symmetry in our apparatus or sample.

Take a moment to examine the apparatus. If we imagine placing a mirror perpendicular to the apparatus and passing through the centers of the two electrodes, we can see that
the image in the mirror would be exactly the same as we see without the mirror. We call this mirror symmetry (or bilateral symmetry) about the $y$-axis because of this fact. Exactly the same stuff is at $(-x, y)$ as is at $(x, y)$ for all $x$ and $y$. Since our apparatus has mirror symmetry about the $y$-axis, we expect that our observations will also have this symmetry. We need to test enough points to convince ourselves that our data is symmetric, but once we are convinced we can simply plot each point at $(-x, y)$ and at $(x, y)$ on the graph paper once its coordinates are determined. If you do not observe mirror symmetry, check for loose nuts, eraser crumbs under your electrodes, or torn Teledeltos paper. Correct any problems before continuing, if possible, or note any complications in your Data. Ask your teaching assistant to help you if you do not see the problem right away.

### Historical Aside

The carbon paper we are using has a trade name: *Teledeltos*. It was developed and patented around 1934 by Western Union. It was originally used to transmit newspaper images over the telegraph lines (as an early fax machine). At the receiving end of the “Wirephoto”, the cylinder of a drum was covered first with a sheet of Teledeltos paper, and then with a sheet of white record paper. A pointed electrode triggered by signals transmitted over the telegraph lines would then reconstruct the image by varying the density of black dots on the record paper.

Now, go after the 4 Volt equipotential using the same technique. Then, do the same for the 3 Volts, 2 Volts, and 1 Volt equipotential lines. For each case, draw a smooth curve among the points having the given potential. Do not just connect the dots to get a segmented line. Remember that our measurements and plots have experimental error in them and that our goal is to average out these errors with a smooth data-fitting curve. The curve is intended to fall along the equipotential between, as well as at, the specific points marked off, so the points should not be connected by straight line segments. Your equipotential lines should look like computer fits to math models. Some data points will be above and some below, but the drawn line will be smooth compared to the data points. Label each line with its potential. It is good strategy to trace the data points in ink and to sketch the lines in pencil until they are satisfactory. This allows you to erase erroneous lines without erasing the data. If you erase pencil lines, please do so away from the apparatus so that the rubber (insulating) crumbs do not degrade its efficiency. Once you are satisfied with the pencil sketches, trace the lines in ink so that the same strategy will apply to the construction of electric field vectors below.

### 4.2.4 Finding electric field lines

Recall Equation (4.3) and apply it to an equipotential line being the path along which the charge moves. Since all points in an equipotential have the same potential, $V_a = V_b$ for equipotentials, the work done as seen from Equation (4.4) is zero, and the work per unit
charge is also zero. Equation (4.3) then becomes

$$0 = (V_a - V_b) = \frac{W_{ab}}{q} = \int_a^b E \cdot dr = \int_a^b E \cdot dr \cdot \cos \theta$$

(4.5)

for points $a$ and $b$ on the same equipotential line, surface, or volume. If the electrodes have different potentials, then $E \neq 0$; an unbalance of charge will make an electric force and field. $dr \neq 0$ unless $a$ and $b$ are the same point; if we moved the charge, then this cannot be. Only $\cos \theta = 0$ or $\theta = 90^\circ$ remains as a possibility, but this means that the electric field, $E$, must be perpendicular to the path that we traveled, the equipotential line. Since we now have a set of equipotential lines, we can use them to sketch the electric field vectors.

**Checkpoint**

In what way are equipotential lines oriented with respect to the electric field lines?

The result discussed earlier that the electric field is everywhere at right angles to the equipotential surfaces and the fact that electric fields start on positive charges and end on negative charges can now be used to draw the field lines in the region where you have traced the equipotentials. On your drawing, place your pencil at a point representing the bar conductor surface and draw a line perpendicular to the bar going toward the nearest equipotential line. As your line approaches the equipotential, be sure that it curves to meet the line at a right angle. Proceed similarly to the next equipotential, and so on until your line ends on the drawing of the round conductor. Keep in mind that each conductor itself is an equipotential, that its surface intersects the paper in an equipotential line, and that the electric field vector must also be perpendicular to these lines. Label the electrode’s images with their electric potentials. Return to the bar in the graph and construct a new line starting at an appropriate distance (say 2 cm) from the first line. Construct 6-8 electric field vector lines. Place an arrow head at the end of each line to indicate the correct direction of the vector.

**Checkpoint**

Can you observe an electric field above and below the paper using this voltmeter? Does the electric field occupy the space above and below the paper? Why can’t this voltmeter observe electric fields in air? How else might these fields be observed?

### 4.3 Finding electric field magnitude

The vectors drawn above are everywhere parallel to the electric field in the paper. Since we know the directions the $E$ vectors point, we can imagine constructing a coordinate axis, $a$, parallel to a segment of one of the field vectors. Figure 4.4 illustrates this process. Let us
consider what Equation (4.3) tells us about this line segment,

\[ V_{ab} = V_a - V_b = \int_a^b \mathbf{E} \cdot \mathrm{d}\mathbf{r} \]
\[ = \int_a^b E \, dr \approx E \int_a^b \mathrm{d}r \]
\[ = E |\mathbf{r}_b - \mathbf{r}_a|. \]  

(4.6)

The electric field is not actually constant in this interval but it also does not change much. Additionally, the Mean Value Theorem of Integral Calculus tells us that at least one point in this interval has electric field equal to this average (in this case exactly one point) and that the value of the integral is equal to the average field times the distance traveled or the interval’s length. Then along the \( a \) axis we have

\[ E_a = E = -\frac{V_b - V_a}{a_b - a_a} = -\frac{\Delta V}{\Delta a}. \]  

(4.7)

**General Information**

Students of calculus might recognize that this (approximate) derivative is the inverse of the integral in Equation 4.3. This relationship is complicated by the three dimensions of the field vectors.

**Checkpoint**

Why must the \( \mathbf{E} \) field be perpendicular to the surface of an ideal conductor?

It turns out that this result can be generalized. The components of the electric field can be calculated or measured using

\[ E_{x_i} = -\frac{\Delta V}{\Delta r_i}, \]  

(4.8)

where \( V(x, y, z) \) is the electric potential function (voltage) as a function of position and the two points whose potential difference we are calculating are parallel to the \( x_i \) axis. Normally, \( x_i \) is either \( x, y, \) or \( z \); we can always find all three components of \( \mathbf{E} \) by finding three potential
differences with one parallel to $x$, another parallel to $y$, and the last parallel to $z$. In this case, however, we have constructed our $a$ axis parallel to $E$ so that $E_a$ is the only component and $E = |E| = E_a$. Additionally, we have not collected potentials at points parallel to $x$, $y$, or $z$, so we do not have the correct information to calculate the field components. For the segment of $E$ between the $2$ V and $3$ V lines, the $2$ V point has a larger value of $a$. In fact, this value of $a$ is larger than $a$ on the $3$ V line by the distance between the lines. Since we know the potentials, we can find the difference. Since we can measure the distance between the lines, we can measure $\Delta a$: $\Delta a$ is the distance between the lines. Actually, the black paper containing the field lines is twice as big as our map and we are measuring $\Delta a$ on our map. To get $\Delta a$ for the black paper, we must double our measurement. Let us suppose that we measure $\Delta a = 2.4$ cm in Figure 4.4. Then the electric field in the black paper has magnitude

$$E_a = -\frac{\Delta V}{\Delta a} = -\frac{2 \text{ V} - 3 \text{ V}}{2(2.4 \text{ cm})} = 0.208 \text{ V/cm.} \quad (4.9)$$

Actually, this is the average electric field strength along this segment of $E$. To get the electric field at a point, we must repeat the experiment again and again increasing the number of equipotential lines each time. We might imagine finding $0.1$ V, $0.2$ V, ..., $5.9$ V equipotential lines and then finding $0.01$ V, $0.02$ V, ..., $5.99$ V, etc. With each repetition the lines get closer together and the average field strength for each segment is closer to all of the points in the segment. In calculus we call this process “taking the limit as $\Delta V$ approaches zero”. Since the lines get closer together as $\Delta V$ decreases, we are also “taking the limit as $\Delta a$ approaches zero”. The ratio, $-\frac{\Delta V}{\Delta a}$, gets closer and closer to some real number that is effectively the value of the field at the point. Symbolically, the components of the electric field at each point in space are given by

$$E_x = -\lim_{\Delta x \to 0} \frac{V(x + \Delta x, y, z) - V(x, y, z)}{\Delta x}$$
$$E_y = -\lim_{\Delta y \to 0} \frac{V(x, y + \Delta y, z) - V(x, y, z)}{\Delta y}$$
$$E_z = -\lim_{\Delta z \to 0} \frac{V(x, y, z + \Delta z) - V(x, y, z)}{\Delta z}$$

Use Equation (4.7) as illustrated in Figure 4.4 to compute the average electric field for all six segments of a single field vector. Mark the vector that you use on your map so that your readers can verify your work.

### 4.4 Analysis

Discuss the properties of the equipotential lines and the electric field vectors. Do these observations have the same symmetry as the apparatus that caused them? What kind of symmetry is this? Does the paper’s potential change when the power supply voltage is changed? These are indications that the apparatus caused the observations.
Were the equipotential lines continuous as predicted? Are the field lines close together at places where the field magnitude is large? Are the equipotential lines more curved at places where the field magnitude is large?

What subtle sources of error are present in this experiment? Are these errors large enough to explain any discrepancies between your observations and the properties of electric fields?

4.5 Conclusions

In science a cause and its effect always have exactly the same symmetry. Can you conclude that your apparatus causes your observed equipotential lines and electric fields? Is Equation (4.7) consistent with our data? If so, include this as part of your Conclusions and define all symbols. Are you confident that this apparatus and method reveals the electric field around these electrodes?

This apparatus has historical significance as a design aid. Experimenters and engineers once constructed electrodes having a particular shape in hopes of obtaining an electric field suited to a specific purpose. For example, we might need to design a vacuum tube to act as an amplifier or we might need to focus an electron beam for use in a TV’s CRT. Today, we can simply download an electrodynamics simulation program to run on our smartphone; but once upon a time the only way we could view the electric field around our electrodes was to measure it using a similar apparatus. What other purposes can you imagine using our apparatus to fill? Motivate interest in our work by pointing out how valuable this tool can be for designing electric fields.
Chapter 5

Experiment 3: Ohm’s ‘Law’

5.1 Introduction

When a potential difference is maintained between the contacts of an incandescent light bulb, the electric field forces charge to flow through the filament of the bulb. The filament resists the flow of charge, and the work done to force the charge through is converted to thermal energy, heating the filament to such a high temperature that it emits light. This light carries away energy. The temperature reached is such that the energy carried away by the light (and conducted away by the bulb holder) exactly balances the electric energy delivered to the filament. The flow of charge is described as an electric current and we would like to explore how the current through a particular material depends on the potential difference applied across the material object.

![Image of electrons drifting in a wire's electric field]

**Figure 5.1:** Sketch of electrons drifting in a wire’s electric field. Moving charge makes electric current $I$.

To examine this question adequately, we must first define quantitatively what is meant by “current” and “resistance”, as well as considering the physical mechanism that accounts for the electrical resistance of a metal. The current passing through a wire is defined by how much charge passes through a cross-sectional area, $A$, of the wire per unit time, as shown in Figure 5.1. You should convince yourself that conservation of charge implies that the current is the same no matter what surface across the wire is used in defining the current. When the current changes in time, however, it is useful to consider the rate of charge crossing the area at each given instant, so that the current, $I$, is defined more generally as

$$I = \frac{dQ}{dt}$$  \hspace{1cm} (5.1)
where \(dQ\) is the charge passing through the area in some very brief time interval, \(dt\), when the observation is made. In the one dimensional case, a positive sign would be assigned to \(I\) when positive charge flows from left to right (positive \(x\)-direction) across the surface, a negative sign when it flows in the opposite direction. The unit of current is a Coulomb per second, which is given the special name “Ampere”, so that 1 Ampere is 1 Coulomb/second or, in official SI symbols, 1 A = 1 C/s.

We are primarily concerned with the current in a metallic conductor. In these systems, the electrons at the higher available energies are reasonably free to migrate from one location to another around the background of positively charged ions of the metal. The electrons moving in this way behave in many respects as if free. The background ions are mostly arranged in an orderly way on lattice sites. But because of the vibrational motion of the ions, and because of impurities and other unavoidable irregularities in the periodicity of the lattice, the electrons are constantly being scattered by the lattice of ions, interchanging energy and momentum with the heavier background ions. This randomizes the electronic motion and keeps the free electrons in thermal equilibrium with the lattice. Using the language of thermodynamics we would say that

\[
\frac{3}{2} M_L \langle v_L^2 \rangle = \frac{3}{2} k_B T = \frac{3}{2} m_e \langle v_e^2 \rangle,
\]

where \(M_L\) is the lattice molecular mass, \(v_L\) is its velocity, \(k_B\) is Boltzmann’s constant, \(T\) is the equilibrium temperature, \(m_e\) is the electron (effective) mass, and \(v_e\) is its velocity; the angular brackets indicate an average over particles and time. When an electric field \(E\) is applied along the conductor (see Figure 5.1), the field imposes a force \(F = -eE\) on the electrons (where \(e\) is the absolute value of the electron charge, \(e = 1.602 \times 10^{-19}\) C). Scattering by the lattice prevents the electrons from accelerating indefinitely, but instead transforms the extra kinetic energy that the electrons acquire from the field into vibrational energy of the lattice. The conductor can become hot (as in the case of an electric stove), or even incandescent (as does the filament of a light bulb). The combined effect of the scattering and the applied field on the electrons is that their average velocity shows a small overall drift velocity, \(v_d\), (typically about 0.001 m/s) in the direction opposite to that of the applied electric field \(E\). This overall drift velocity of the electrons, superposed on their random thermal motion, is responsible for the electric current.

While the current in a metallic conductor is caused by the motion of electrons, there are systems that have moving positive charges or even both positive and negative charges moving simultaneously in opposite directions. A plasma is a gas of ionized particles in which the particles have had one or more electrons separated from a positive core or nucleus. The current in plasmas is due to the nuclei moving parallel to the electric field and the electrons moving opposite to the field. Conductive liquids, like the electrolyte in a car battery, have ionized atoms and/or ionized clusters of atoms. The electric current results from positive and negative ions moving in opposite directions. Semiconductors have electrons broken away from a chemical bond by thermal vibrations that then move about the crystal. The vibration leaves behind a positive ‘absence of an electron’ or hole that can be filled by borrowing an electron from a nearby bond. Semiconductor currents have holes moving parallel to the field and electrons moving opposite to the field. Hydrogen fuel cells have positive hydrogen ions moving into a membrane on one side and negative oxygen ions moving into the membrane.
on the other side (and in the opposite direction). The ions combine in the membrane to form neutral water.

**Historical Aside**

Benjamin Franklin made significant contributions to the understanding of electricity by proposing a “single fluid theory” in which electrification by friction resulted from transferring particles of an “electrical fluid” (or “charge” from the modern viewpoint) from one object to the other. Objects with a deficiency of this fluid were negatively charged, and those with an excess would be positively charged. But, lacking our present knowledge of electrical phenomena, Franklin assigned a negative charge to the amber that had been rubbed with fur. This unfortunate choice was seen eventually to require assigning a negative charge to the electron, thus complicating the lives of future generations of physics students in their efforts to understand which way the charge ‘really’ flows in an electric circuit.

By convention, the direction of the electric current is taken to correspond to the flow of *positive* charges in the direction of the applied electric field from the higher to the lower electrical potential even when the actual charge carriers are negatively charged electrons which, in reality must move in the opposite direction (see Figure 5.1 and Figure 5.2). Sometimes, for emphasis, the term “conventional current” is used to distinguish the electric current defined in this way from the *particle current* motion of electrons. This seemingly artificial sign convention does not affect the validity of the equations based on it. The results are the same whether the current is taken to be that of positive charges flowing from a positive potential to a negative potential, or that of negative charge flowing from the negative potential to a positive potential. Indeed, close examination of the definition of current for the one-dimensional case, Equation (5.1), shows that a flow of negative charge to the left produces a positive current just as if positive charge really were flowing to the right.

**Checkpoint**

What is an electric current?
Only the Hall effect can distinguish positive charge motion from negative charge motion mathematically. Later you will be able to understand how. When an electric field is constructed perpendicular to a magnetic field, the electric field accelerates positive and negative charges in opposite directions. Because of their opposite movement and their opposite charges, both kinds of charged particles collect on the same side of the object. This side becomes positively charged by motion of positive charge carriers and negatively charged by negative carriers. We can tell the sign of the dominant charge carriers by noting the potential on this surface.

5.1.1 The Electromotive Force

Maintaining a steady current in a wire requires a source of electrical energy, such as an electric battery or electric generator (see Figure 5.2). Chemical energy in the battery or mechanical energy in the case of the generator, is converted into electrical energy by doing work on the charges passing through. This “electromotive force” (or “emf”) is the work done per unit charge by the battery or generator to move charge from lower to higher potential. The units of emf are Volts (V). For a battery, the emf is also equal to the voltage across the battery terminals when nothing is connected to the battery terminals and no significant current flows. Note that the customary term “electromotive force” is somewhat misleading since the emf is an electric potential difference and not a force.

Checkpoint

What is an emf?

5.1.2 Resistance and Ohm’s Law

Early in the nineteenth century a German mathematician and physicist named Georg Simon Ohm discovered that, as long as the temperature was kept constant the magnitude of the current in a metal is proportional to the applied voltage as shown in Figure 5.3. This relationship is now known as Ohm’s law. Ohm defined a property of the conductor that he called the resistance, \( R \), to be the proportionality constant in this law,

\[
R = \frac{V}{I} \quad \text{or} \quad V = IR .
\]  

(5.2)

Resistance resists the flow of charge; a larger resistance requires a larger emf or potential difference, \( V \), to result in the same current, \( I \). The units of resistance is the Ohm (1 \( \Omega = 1 \) V/A) in Georg’s honor.

The resistance of a particular conductor or resistor depends on the physical dimensions of the resistor material and the resistivity property of the material itself. If a resistor has
length, $L$, uniform cross-sectional area, $A$, and resistivity, $\rho$, then its resistance will be

$$R = \rho \frac{L}{A}. \quad (5.3)$$

The properties of materials, including their resistivity, tend to change as thermodynamic attributes such as temperature changes. In the particular case of resistivity, the change with temperature is fairly linear over a wide range of temperatures for most materials. The slope of this line is the temperature coefficient of resistivity, $\alpha$, and also is a property of materials. We model the temperature dependence of resistivity using

$$\rho(T) = \rho_0 [1 + \alpha (T - T_0)] \quad (5.4)$$

where $\rho_0$ is the measured resistivity at temperature $T_0$. The resistance of most conductive materials (and all metals in particular) increases with temperature increases so that $\alpha > 0$ for most conductive materials. The resistivity of insulators and semiconductors, on the other hand, tends to decrease with increased temperature and $\alpha < 0$.

Since vibrations of the ions around which the electrons move typically produce most of the scattering that impedes the response of the electrons to an applied field, and since the amplitude of these vibrations increases with temperature, the resistance of most solids tends to increase with temperature as shown in Figure 5.3. But thermal vibrations can also break electrons free of chemical bonds and thus increase the number of moving charges and the current. In insulators and semiconductors that have very few free charges, freeing charges in this way decreases resistance more than the increased scattering increases it.

Figure 5.3: Temperature dependence of resistivity for several metals. The low-temperature dependence is in the inset.

Checkpoint

Which way does the current flow when the applied electric field is from left to right along a wire? Which way do the electrons flow?

Checkpoint

State Ohm’s Law.
Anything that has electrical resistance as its primary electrical response is termed a resistor. In a circuit diagram, the symbol for a resistor is a zigzag line (see Figure 5.4). Everything (except superconductors) has some electrical resistance and we include a resistor symbol in its model when we wish to emphasize this fact. The resistors used in most applications are “carbon composition” resistors - small cylinders with wire leads at each end, constructed of powdered graphite in clay. The resistivity can be controlled using the ratio of graphite to clay. Most resistors are color-coded with a series of colored bands on them to indicate their designed resistance. Figure 5.4 and Table 5.1 detail how this color code is defined and used.

The color coding of an 820 Ohm resistor with 5% tolerance. The first two bands specify an integer between 10 and 99. The third band gives a power of ten multiplier (how many zeroes follow the integer). The fourth band specifies the tolerance: 5% if gold, 10% if silver, 20% if omitted. The first two bands define the significant digits or mantissa of the scientific notation. The ten possible colors correspond to the ten decimal digits 0-9. The third band indicates the exponent of the scientific notation or the number of zeros that need to be added after the two digits. In addition to the normal ten digits, the exponent band can also be silver or gold to allow resistances less than $10\, \Omega$. A silver exponent band means move the decimal point two places to the left ($0.10\, \Omega - 0.91\, \Omega$) and a gold band means move it one place to the left ($1.0\, \Omega - 9.1\, \Omega$). The last band, which is gold, silver, or simply absent, denotes...
the manufacturer’s tolerance in achieving the indicated value of resistance. Nothing can be made arbitrarily well. The molds the manufacturer uses to form the composite carbon and clay have error tolerances in their area and length. The composite that fills the molds has tolerances for the ratio of graphite to clay as well as for any impurities that cannot be eliminated from the graphite and clay. The manufacturer has tolerances for how much composite material gets packed into each mold. But all together the manufacturer is confident that his product will maintain its resistance within the specified tolerance for all temperatures and humidities that do not damage the device. The fourth band summarizes the specified tolerance: gold ⇔ 5%, silver ⇔ 10%, and no band means 20%. If a fifth band is present, it specifies the product reliability or failure rate. An example of using this coding scheme is shown and explained in Figure 5.4.

Table 5.1: The decimal value definitions for colors in the industry standard resistor color code.

<table>
<thead>
<tr>
<th>Color</th>
<th>Value</th>
<th>Tolerance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Brown</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Red</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Orange</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Yellow</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Green</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Blue</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Violet</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Grey</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>White</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>Gold</td>
<td>−2</td>
<td>5%</td>
</tr>
<tr>
<td>Silver</td>
<td>−1</td>
<td>10%</td>
</tr>
<tr>
<td>none</td>
<td></td>
<td>20%</td>
</tr>
</tbody>
</table>

The work done on a charge, \( q \), by an electric field, \( \mathbf{E} \), is

\[
W = \int_{a}^{b} \mathbf{F} \cdot d\mathbf{r} = q \int_{a}^{b} \mathbf{E} \cdot d\mathbf{r} = qV_{ab}. \tag{5.5}
\]

To compute the power expended by the electric field in moving a current through a resistor, we need the rate that work is performed. Since the potential difference in this case is independent of time,

\[
P = \frac{dW}{dt} = \frac{dq}{dt} V = IV. \tag{5.6}
\]

We can use Ohm’s law to substitute for \( I \) or for \( V \) to find three equivalent forms of Joule’s law of electrical heating,

\[
P = IV = I^2R = \frac{V^2}{R}, \tag{5.7}
\]

and dissipated power has units of Volt-Amperes or Watts after James Watt who invented the steam engine and several thermodynamics facts. The units of Volts and Amperes have been defined in a way that relates electrical and mechanical quantities, because a Volt is an amount of work per Coulomb, so that

\[1 \text{ Volt-Coulomb} = 1 \text{ Volt-Ampere-second} = 1 \text{ Watt-second} = 1 \text{ Joule} = 1 \text{ Newton-meter}.
\]

**Checkpoint**

By what factor does the power increase when the current through a resistance is doubled?
CHAPTER 5: EXPERIMENT 3

**Electrical Circuits and Circuit Diagrams**

We illustrate circuits using *schematic diagrams* whose components are depicted with symbols. These symbols play the same role in our study of circuits as the alphabet plays in written language or the ten numerals play in our number system. Figure 5.5 is one example of the schematic diagram. We might contrast this with the ‘connection diagram’ in Figure 5.6 or the apparatus’ photograph on the lab’s website. The schematic diagram contains all information needed to construct the circuit from arbitrary components. Many different corporations manufacture resistors, voltmeters, ammeters, and power supplies and it is thus unlikely that our readers each have exactly our combination; however, if he chooses he can reproduce our work using the components that he has in his lab.

**5.1.3 Equipment**

The plug-in boards we will use in this experiment allow you to assemble a circuit quickly from component parts. In this lab we will assemble a simple circuit with a power source (emf) and a single resistor. We will include in the circuit a means of measuring the voltage of the emf and the corresponding current that this emf drives through the resistance as we vary the emf. We will do this experiment for each one of three resistances with values of 1 kΩ, 470 Ω and 100 Ω, for a light bulb, and for a light emitting diode (LED). To prepare for the lab you should calculate the currents through resistances of the values above for a supply voltage of 6 Volts maximum. The resistances you will use are rated for 2.0 Watts. Also, compute the power each resistor will dissipate and check to make sure this power rating is not exceeded for any of the resistances you will be using BEFORE you power up the circuit for real.

Pasco’s 850 Universal Interface will provide a variable voltage power source that you can control from their Capstone computer program. The voltage output leads can be located on the right top of the 850 Interface box’s front panel. Connect the positive and negative terminals to the appropriate sockets on the plug-in board. The negative output is indicated with a ground symbol (see Figure 5.4) and the positive output is indicated with a sine wave. Construct a simple circuit using the 100 Ω resistor as shown in Figure 5.6.

The voltage and current will be measured using Pasco’s 850 Universal Interface and Capstone program. Connect the voltage leads to input A on the 850 Interface box. Apply the leads to the ends of the circuit element to be tested, as shown in Figure 5.6 for the case...
Figure 5.6: Sketch of an electric circuit constructed on our plug-in breadboards. The circuit consists of the Interface’s output providing an emf, an ammeter to allow the current to be measured, and a resistor to limit the flow of current. External to the circuit and yet essential is the voltage sensor in parallel with the resistor.

You can begin and stop collecting data by clicking the “Record” button at the bottom left. The button then changes into a “Stop” button so that clicking it again ends the collection run.

Click the “Signal Generator” at the left to bring up the power supply control. It should already be configured to supply a constant potential difference and increasing or decreasing the output level will vary the particular potential difference generated. Set the emf you desire and click “Record”. Verify that both the resistor current and the resistor voltage are constant and click “Stop”. Small variations in the current and voltage readings suggest the 850’s uncertainties in these measurements. Record the voltage and current in a data table.
or enter them directly into Vernier Software’s Graphical Analysis 3.4 (Ga3); a suitable setup for Ga3 is also supplied on the website. Estimate the uncertainties and enter these into your data table as well.

Now choose another applied voltage and “Record”, observe, and “Stop” again to generate another \((I, V)\) data point for your data table (and graph). Select about 10 different applied voltages between -5 V and +5 V for each component. Do your data points lie along a line in your graph? If they do, draw a box around the data points, “Analyze/Curve Fit...”, and choose “Ohm’s Law” from near the bottom of the list. “Try Fit” and “Done” to fit them to Ohm’s law. If they do not lie along a line, it makes no sense to model them as a line; can you think of another model that might work? If not, just display the data points. If your fit parameters do not include the uncertainties, right-click the box, “Properties...”, and select “Show Uncertainties”. Print the data table and the graph for your report.

### Helpful Tip

If you want to use a Word processor, you can select the table and then the graph and copy each in turn directly into the Word report. Don’t forget to generate a label (Caption) so that you can talk about it in your main text.

A suitable Word template is also provided on the website to get you started.

### Checkpoint

Why must the voltage be read directly across the material being tested rather than using the voltage reading of the source?

### Checkpoint

Why must the Ammeter interrupt the circuit and not be placed in parallel with the element through which the current is being measured?

Repeat the experiment for the other two resistors. Repeat for the incandescent lamp; is the lamp data linear? If not, it would be counterproductive to fit it to a line. Plot current on the \(x\)-axis and voltage on the \(y\)-axis by clicking on the axis labels in turn. Does the graph have odd symmetry? (Is \(f(-x) = -f(x)\)?) If so, the function has only odd powers and we can model it using

\[
V(I) = R_0 I + R_2 I^3 + \ldots \tag{5.8}
\]

Data/Sort Data using current. Drag a box around your data points and Analyze/Curve Fit... Choose the “Odd Cubic” model near the bottom of the list.

Why is the lamp data not linear? Have you made observations to support this? Note
them in your Data. Be as detailed in your explanation as you know how to be. Offer all evidence in support that you remember observing. Feel free to gather more data so that you can pay closer attention to details.

Repeat once more using the LED. Is the LED’s data linear? Does the LED’s data have odd symmetry? Even symmetry? No symmetry at all? Is there any reason why this data should have more error than the data from the other samples (resistors or lamp)?

**Checkpoint**

What is meant by the term “ohmic material”?

### 5.2 Analysis

Use the strategy in Section 2.9.1 to decide whether each of your measured linear slopes is the same as the respective manufacturer’s specified resistance. It might be necessary for you to convert the units on one of these two numbers before it is valid to compare them. What other subtle experimental errors are present in your measurements? Might some of these be large enough to explain any additional disagreement?

Discuss the lamp’s data. Is current directly proportional to voltage as Ohm predicted? Might we correct for resistance changes to make the lamp’s voltage proportional to current at each data point? What is the cause of these resistance changes? Keep in mind that the lamp’s filament is the metal tungsten (W) and describe in detail how increasing voltage can result in resistance increases.

Discuss the LED’s data. Is current directly proportional to voltage as Ohm predicted? Do you even get the same absolute value of current for positive and negative voltages? Does the LED get hot? Light it and touch it to be sure. Do LEDs obey Ohm’s law? Is Ohm’s “law” a law or does it have exceptions? Can you prove your answer?

### 5.3 Conclusions

How well do resistors obey Ohm’s law? Might Ohm’s law be useful to electrical engineers? Consider including your measured resistance values and units. Discuss what your lamp says about Ohm’s law and correcting resistance for temperature changes. Do you have a model (equation number) that fits your lamp data? Discuss what your LED says about Ohm’s law. Is Ohm’s law a law?

Always communicate with complete sentences; “Yes” is a very poor answer for these suggested discussion topics and equations are not sentences but may be used as nouns. How might we improve our experiment? Do you see any applications for what you have observed?
Chapter 6

Experiment 4:
Electric Currents and Circuits

6.1 Introduction

The resistance to the flow of an electric current is essential in the design of electronic devices and electric circuits generally. We study in this lab session how the electric current distributes itself when alternative competing paths for the electric current offer different resistances to the flow of charge.

6.1.1 The Electric Current

In a previous lab we learned that the current passing through a wire is defined by how much charge passes through a cross-sectional area, $A$, of the wire per unit time. By convention, the direction of the electric “conventional current” is taken to correspond to the flow of positive charges, in the direction of the applied electric field from the higher to the lower electrical potential, even when the actual charge carriers are negatively charged electrons, which in reality must move in the opposite direction.

In a previous lab we learned about Ohm’s law. Ohm defined the resistance, $R$, of a conductor as the proportionality constant in this law, given by the voltage divided by the current

$$V = IR.$$  (6.1)

Anything that has electrical resistance may be termed a resistor. In a circuit diagram, the schematic symbol for a resistor is a zigzag line (see Figure 5.4).

The resistors used in most applications are called “carbon composition” resistors - small cylinders with wire leads at each end, constructed of powdered graphite mixed in clay and compressed into molds. Most resistors are color-coded with a series of bands on them, starting from closest to the edge as shown in Figure 5.4 and Table 5.1. The last band, which is gold, silver, or simply absent, denotes the manufacturer’s tolerance in achieving the indicated value of resistance. The band just before that indicates a power of ten multiplying the previously coded numbers. The remaining colors, starting from the edge, indicate a
sequence of digits. An example of this coding scheme is shown and explained in Figure 5.4.

To compute the power expended in moving a current flowing in a resistor, we note that the work the electric field does on a charge \( dq \) that “falls” through a potential difference \( V \) is \( V \, dq \). Hence, the work done per second is

\[
P = V \frac{dq}{dt} = VI = I^2R = \frac{V^2}{R}
\]

and is in units of Volt-Amperes or Watts. The units of Volts (V) and Amperes (A) have been defined in a way that relates electrical and mechanical quantities, because a Volt is an amount of work per Coulomb (C), so that

\[
1 \text{ Volt-Coulomb} = 1 \text{ Joule} = 1 \text{ Watt-second} = 1 \text{ Newton-meter}
\]

The conservation of electric charge lead the German physicist Gustav Kirchoff to note that every circuit node must have

\[
0 = \sum_{i=1}^{N} I_i
\]  

(6.2)

where the \( N \) branches or current paths connected to the node have currents \( I_i \) entering the node; currents exiting the node have negative signs. Since charges are neither created nor destroyed, every charge that enters a node (+) must also exit (-). Entering it adds its charge and exiting it subtracts its charge for a zero sum.

\[ (a) \]

\[ (b) \]

\[ \begin{array}{c}
V_T \\
A \\
I_T \\
R_1 \\
A \\
I_1 \\
R_2 \\
A \\
I_2 \\
R_3 \\
A \\
I_3
\end{array} \]

\[ \begin{array}{c}
V_1 \\
R_1 \\
V_2 \\
R_2 \\
V_3 \\
R_3 \\
V_T
\end{array} \]

\[ (a) \quad (b) \]

**Figure 6.1:** Examples of series and parallel circuits containing three resistors and source of emf.

The fact that the electric force is conservative led Kirchoff to realize that every closed loop must have

\[
0 = \sum_{i=1}^{N} V_i
\]  

(6.3)

where element \( i \) causes an increase (+) or a decrease (-), \( V_i \), in electric potential. This is because the work done by conservative forces depends only upon the endpoints of the path.
If the two endpoints are the same point, as is necessary for completing a loop, then the work done must vanish. If we take any charge, \( q \), and carry it around the closed loop under discussion, the work is described by multiplying Equation (6.3) on both sides by \( q \).

These relations are called Kirchoff’s node (or current) rule and Kirchoff’s loop (or voltage) rule, respectively. They are irreplaceable in the analysis of circuits as the following discussion will indicate.

The total resistance, \( R_T \), of a complex combination of interconnected resistances is obtained by dividing the voltage applied across the combination by the total current flowing through the combination. Two special cases are the series circuit and parallel circuit (Figure 6.1). In each of the resistance combinations shown in Figure 6.1, the total resistance is \( R_T = \frac{V_B}{I_B} \) where the subscript refers to the element (‘battery’ in this case). For the series circuit, the same current, \( I = I_B \), flows through each resistor and the battery, but the potential difference across the entire series \( V_B = I_B R_T \) is equal to the sum of individual voltage changes \( V_i = I_B R_i \) across each resistor. This equality directly implies that for resistors in series

\[
R_T = \frac{V_B}{I_B} = \frac{1}{I_B} \sum_{i=1}^{N} V_i = \frac{1}{I_B} \sum_{i=1}^{N} I_B R_i = \sum_{i=1}^{N} R_i
\]  

(6.4)

where the battery increases the potential, the resistors decrease the potential, and we have used Kirchoff’s loop rule around the loop of battery and series resistors.

For the parallel circuit in Figure 6.1, it is the voltage across each resistor that is the same, but now the current, \( I_B \), is distributed among different paths, so that the total current \( I_B = \frac{V_B}{R_T} \) is the sum of currents \( I_i = \frac{V_B}{R_i} \) through each branch. This leads immediately to

\[
\frac{1}{R_T} = \frac{I_B}{V_B} = \frac{1}{V_B} \sum_{i=1}^{N} I_i = \frac{1}{V_B} \sum_{i=1}^{N} \frac{V_B}{R_i} = \sum_{i=1}^{N} \frac{1}{R_i}
\]  

(6.5)

where the battery current enters the positive node, the resistor currents exit the node, and we have applied Kirchoff’s node rule to the positive node.

For many complicated circuits, the resistance can be found by breaking the circuit down into combinations of resistances all in series and resistances all in parallel, using Equation (6.4) and Equation (6.5) to calculate the resistance of each branch and then of each required combination of branches, as illustrated at the end of the example given below. Sometimes also, as in this lab session, the voltages between different points in the circuit can be measured, and Equation (6.4) and Equation (6.5) together with Ohm’s law, Equation (6.1), can be applied to different branches of the circuit to find the current in each.

**Checkpoint**

How do the values of resistors connected in series combine? How do the values of resistances connected in parallel combine?
CHAPTER 6: EXPERIMENT 4

Checkpoints

When two resistors are connected in parallel, and the resistance of one of them is increased, how does the overall resistance of the circuit change? Why?

When two resistors are connected in series, and the resistance of one of them is decreased, how does the overall resistance change?

6.1.2 Example of Circuits with Resistors

The following example illustrates how to calculate the current through different parts of a complicated circuit. A 6 V battery is connected to the circuit shown in Figure 6.2. The known resistances are $R_1 = 100 \, \Omega$ and $R_3 = 1 \, k\Omega$. You are given a voltmeter and must determine the currents $i_1$, $i_2$, and $i_3$ as well as the unknown resistance $R_2$. You could measure the voltage $V_1$ across $R_1$ and you could use Ohm’s law to calculate $i_1$. Assume you measured a value of $V_1 = 2 \, \text{V}$ (i.e., 2 Volts). Then

$$i_1 = \frac{V_1}{R_1} = \frac{2 \, \text{V}}{100 \, \Omega} = 20 \, \text{mA}.$$  

The total voltage ($V_B$) across the battery is equal to the sum of the voltages $V_1$ across $R_1$ and the voltage (which we call $V_{23}$) across $R_2$ and $R_3$. Then

$$V_B = V_1 + V_{23}$$

and

$$V_{23} = V_B - V_1 = 6 \, \text{V} - 2 \, \text{V} = 4 \, \text{V}.$$  

(You could have measured this voltage with the voltmeter.) Ohm’s law now allows us to calculate $i_3$ because we have just calculated the voltage across $R_3$ and because the resistance is known,

$$i_3 = \frac{V_{23}}{R_3} = \frac{4 \, \text{V}}{1 \, \text{k}\Omega} = 4 \, \text{mA}.$$  

To determine $i_2$ we use the principle of conservation of charge, which implies

$$i_2 = i_1 - i_3 = 20 \, \text{mA} - 4 \, \text{mA} = 16 \, \text{mA}.$$  

Finally, to determine $R_2$ we use Ohm’s law once more

$$R_2 = \frac{V_{23}}{i_2} = \frac{4 \, \text{V}}{16 \, \text{mA}} = 250 \, \Omega.$$
As a check, we can calculate the total current the battery must supply. First calculate the resistance $R_{23}$ of the branch containing $R_2$ and $R_3$

$$\frac{1}{R_{23}} = \frac{1}{R_2} + \frac{1}{R_3} \quad \text{or} \quad R_{23} = \frac{R_2 R_3}{R_2 + R_3}.$$ 

Then calculate the overall resistance $R_T$ of the circuit as

$$R_T = R_1 + R_{23} = 100 \, \Omega + \frac{(250 \, \Omega)(1000 \, \Omega)}{250 \, \Omega + 1000 \, \Omega} = 300 \, \Omega$$

and find the current using

$$I_B = \frac{V_B}{R_T} = \frac{6 \, \text{V}}{300 \, \Omega} = 0.02 \, \text{A}.$$ 

This is the current previously found to be flowing first through $R_1$ and then branching to either $R_2$ or $R_3$.

The power dissipated in a resistor is given by $P = I^2R$, so that the power dissipated in $R_1$ is

$$P_1 = I_1^2 R_1 = (0.02 \, \text{A})^2(100 \, \Omega) = 0.04 \, \text{W}.$$ 

In the second resistor, the power dissipated is

$$P_2 = I_2^2 R_2 = (0.016 \, \text{A})^2(250 \, \Omega) = 0.064 \, \text{W},$$ 

and in the third resistor,

$$P_3 = I_3^2 R_3 = (0.004 \, \text{A})^2(1000 \, \Omega) = 0.016 \, \text{W}.$$ 

The total power expended maintaining the current is then

$$P = P_1 + P_2 + P_3 = 0.12 \, \text{W}.$$ 

To check the calculated results, we can instead evaluate the total power directly from the total current $I_B = 0.02 \, \text{A}$ through the battery and the potential difference $V_B = 6 \, \text{V}$ across its terminals

$$P = V_B I_B = (6 \, \text{V})(0.02 \, \text{A}) = 0.12 \, \text{W}.$$ 

This agrees with the previous value obtained.

**Checkpoint**

By what factor does the power increase when the current through a resistance is doubled? Which branch of a parallel circuit consisting of two unequal resistors in parallel gets the greater current?


## Checkpoint

In a series circuit containing two unequal resistances which resistance has the greater voltage drop across it?

### 6.2 Apparatus

The plug-in breadboards we will use in this experiment allow you to assemble your own resistance network to investigate the electronic properties and principles of parallel (Figure 6.1(a)) and series (Figure 6.1(b)) resistance circuits. In this lab we will assemble a circuit of each type using one each of resistances with values of 1 kΩ, 470 Ω, and 100 Ω. To prepare for the lab you should calculate the voltage across each of the series resistors and currents through each of the parallel resistors for a supply voltage of 6 Volts. The resistances you will use are rated for 2.0 Watts. Check to make sure this power rating is not exceeded for any arrangement of resistances you will be using BEFORE you power up the circuit for real.

#### 6.2.1 The Parallel Circuit

Assemble the circuit on the plug-in board using resistances of 1 kΩ, 470 Ω, and 100 Ω in parallel and shorting jumpers (jumpers are plug-in connecting wires with negligible resistance used to interconnect circuit elements) as shown in Figure 6.3. Each of the five sockets in the array connected by a cross is internally connected with conductors. Circuit elements can be interconnected by bridging across the gaps between each of these arrays of sockets. The Pasco 850 Hardware Interface contains a variable power source that we will set to 6.00 V. The jumpers to the right of each resistor may seem unnecessary but will have a function. We will be measuring the current through each of the branches of the circuit as well as the voltage drop across each resistance element. The voltages and currents will be measured using 850 Interface sensors. The 850 Interface will report its readings to the computer via the USB protocol and the computer will display the results on the video monitor.

### WARNING

You should not attempt to measure the current directly from the power source. Do not connect the leads of the ammeter across the power supply terminals or any leads coming from the power supply. Doing so will blow the inline fuse and cause the ammeter to malfunction. It is also possible that the fuse will not protect the meter and that the current sensor will be destroyed entirely. If you have any doubts about your connection, have your Lab Instructor look over your connections before you turn on the power.

Click the ‘Record’ icon at bottom left to begin gathering data. Now place one voltmeter lead in each of the two places you want to know the potential difference between. Begin by
placing the leads across the power supply and recording the ‘Total’ voltage $V_T$. Record the reading in your data table exactly as the meter displays it. What is your experimental error and units? Be sure to note these as well. *Once this is recorded do not change it again* or all of your voltages and currents will also change to the new value instead of the one you recorded. Now measure and record all three resistor voltages by placing one meter lead on each side of the relevant resistor.

To measure the current in a branch, the circuit must be broken and an Ammeter inserted as shown in Figure 6.3. The ammeter cannot know what the current is unless all of the current passes through the meter. In Figure 6.3 the jumper in the net circuit resistance is removed and the leads to the ammeter are inserted as shown to measure the current through the entire circuit. Simply ignore any negative sign; reversing the leads will eliminate it but the absolute value will be the same in either orientation. Once the current provided by the ‘battery’ is measured, replace the shunt and measure the current in turn through the three resistors in the same way.

Compare the total current with the sum of the three branch currents. If this sum is significantly different from the battery current, you are probably measuring the currents
wrong. Calculate measured values of resistance for all three resistors and for the total circuit resistance using

\[ R_i = \frac{V_i}{I_i} \]  

(6.6)

where \( i = 1, 2, 3, \) and \( T \). Use the manufacturer’s specified resistances and Equation (6.5) to predict the total circuit resistance. Use your measured resistor currents to predict your battery current using Kirchoff’s node rule,

\[ I_T = I_1 + I_2 + I_3. \]  

(6.7)

Helpful Tip

It would be a good idea to start a “Results” table to hold these calculated predictions. You can add the results from the other two circuits to this table as they become available.
Figure 6.5: Sketch of our apparatus used to construct a series parallel combination circuit. The voltmeter is set to measure the voltage across the parallel lamps. The ammeter measures the battery current.

6.2.2 The Series Circuit

Modify your parallel circuit so that the resistances form a series circuit as shown in Figure 6.4. Note the placement of jumpers and power supply connections. Does the circuit match the series circuit in Figure 6.1(b)? Use the voltage sensor to record the voltages across each of the resistors in the circuit. In Figure 6.4 we show the connections to measure the voltage across the first resistor. Measure the voltage across the battery. Record these voltages, their errors, and their units in your series circuit Data table. Add the three resistor voltages and compare the sum to the battery voltage. If these are significantly different, you are probably measuring the voltages wrong.

Use the current sensor to measure the current being drawn from the battery; Figure 6.4 shows how to do this. Remove the meter leads and replace the meter with a shunt. Remove
one of the shunts and place the ammeter leads in its place to connect the meter in series with the resistor. Record the resistor’s current and replace the shunt. Repeat this to record the other two resistor currents. Note the experimental errors and units in your Data table also.

Use the battery voltage and the battery current to compute a measured total circuit resistance. Use the resistor values measured in “The Parallel Circuit” section and Equation (6.4) to predict the total circuit resistance. Use your measured resistor voltages and Kirchhoff’s loop rule to predict the battery voltage,

\[ V_B = V_1 + V_2 + V_3, \] (6.8)

and record these predictions in your table of results.

6.2.3 The Series-Parallel Circuit

We would like to explore a circuit that uses both series and parallel connections. The circuit shown in Figure 6.2 is a most useful circuit. For example, the electrical power lines have resistance, \( R_1 \), and the lights, \( R_2 \), and air conditioner, \( R_3 \), in your home are each powered by them. When your air conditioner first switches on, your lights dim due to the transient current needed to start the motors turning. Assemble the circuit using the plug-in board and jumpers. For each resistance use a socketed incandescent lamp. Ask your lab instructor to verify the circuit before you continue. We would like to measure the effect of adding the \( R_2 \) resistance in parallel to the \( R_3 \) resistor, which is connected in series with the \( R_1 \) resistor. Power up the circuit and note what happens when \( R_2 \) or \( R_3 \) is removed and re-inserted.

BEFORE you come to lab, assume that the resistances of the three identical light bulbs are equal and compute \( V_1 \) and \( V_2 = V_3 \) if the battery is 6V.

Use the voltage sensor to measure all three lamp voltages and the power supply voltage and record all four in your series-parallel Data table. Note your experimental errors and units. Check that \( V_B \approx V_1 + V_2 \approx V_1 + V_3 \); if not you are probably measuring the voltages wrong. How well do your measurements agree with your pre-lab calculations?

Substitute the ammeter for the shunts one at a time and record the lamp currents. Remove the meter leads from the circuit and replace all of the shunts. Remove the battery current shunt and measure the battery current.

Can you think of a real circuit of which this might be representative? There are many! Here is a partial list:

- Power transmission wires have resistance and power a house with motors and lights wired in parallel.
- A voltage divider under load.
- A power supply with internal resistance powering multiple devices.
- Automobile headlights in parallel with starter motor; these are powered by wire and battery with internal resistance.
Use Ohm’s law and your current and voltage measurements to compute measured values for all three lamp resistances and the total circuit resistance; record this in your Data table. Predict the value of $I_B = I_1$ using $I_B = I_2 + I_3$. Predict the value of $V_B$ using $V_B = V_1 + V_2$ or $V_B = V_1 + V_3$. Predict the value of $R_{23}$ using $\frac{1}{R_{23}} = \frac{1}{R_2} + \frac{1}{R_3}$ and your measured parallel lamp resistances. Predict the value of $R_T$ using this equivalent parallel resistance, the measured series lamp resistance, and $R_T = R_1 + R_{23}$. Record these predictions in your table of results.

Are the three ‘identical’ lamp resistances the same? Can you explain any discrepancies? Are all three very different or are some similar and the other very different? Might this be a clue to understanding the discrepancies? If you think the lamps are different after all, try swapping the different lamp with one of the similar lamps and repeating the measurements of voltage. Are these measurements significantly different than the previous ones? If so, the lamps might indeed be different. Which lamps are brightest now? What might the brightness indicate about the lamps’ resistances.

**Checkpoint**

Which branch of a parallel circuit consisting of two unequal resistors in parallel dissipates more power?

**Checkpoint**

In a series circuit containing two unequal resistances which resistance dissipates more power?

### 6.3 Analysis

Use the strategy in Section 2.9.1 to decide whether your data supports the equivalent resistance formulas and/or Kirchoff’s rules. You will need to do this seven times: 1) parallel current, 2) parallel resistance, 3) series voltage, 4) series resistance, 5) S-P current, 6) S-P voltage, and 7) S-P resistance. Do your predictions agree with your measurements within the 5% resistor tolerances? What other subtle sources of error might have affected your measurements. Might some of these be large enough to explain the remaining disagreement?

Discuss your series-parallel lamp experiment. What supplemental observations have you made (in addition to meter readings) that help explain your data? Refer to your Ohm’s Law report and see if you can prove what is causing the lamps’ resistances to be different.
6.4 Conclusions

Enumerate the four new physical relationships that your data support. (If you have numbered your equations, you may reference relevant ones here instead of re-writing them.) Define all symbols; symbols used in several relations need defining only once. The first circuit had three resistors in parallel \((N = 3)\) and the last circuit had two resistors in parallel \((N = 2)\). The second circuit had three resistors in series \((N = 3)\) and the last circuit had two resistors (one was the parallel combination of two lamps) in series \((N = 2)\). Strictly speaking, we have not tested cases where \(N > 3\), but the same theory was used to model these cases as for \(N < 4\); we might expect, therefore, that the outcomes would be the same when we eventually do check these cases. The first two circuits contained only linear resistances (see Ohm’s law data) and the last circuit contained nonlinear lamps. The theory works in all tested cases. What would you expect for circuits containing LEDs, transistors, and photodiodes that we also haven’t yet tested? The theory works when we have only parallel resistors, only series resistors, and mixed up combinations of series and parallel resistors. We have demonstrated all of this with only three circuits...

What improvements might you suggest for the experiment? Do you see any potential applications for what you have observed?
Chapter 7

Experiment 5: RC Circuits

7.1 Introduction

A capacitor, often referred to as a condenser, is a simple electrical device consisting of two nearby conducting surfaces separated by an insulator. The capacitor can store charge of opposite sign on the two plates, and is of immense importance in the design of electronic devices. The purpose of this lab session is to examine how a capacitor stores its electric charge and how it discharges by producing a current through a resistor.

Historical Aside

The ability of a capacitor to store charge was discovered accidentally in two independent experiments at different places during the same year. The first discovery was by Ewald Georg von Kleist in October of 1745 when he tried using an electrostatic generator to place a charge on an iron nail inside a small glass bottle. Later that same year, Anreas Cuneus, a lawyer who frequently visited one of the laboratories at the University of Leiden, was trying to electrify water. He used a chain hanging into a flask of water, and brought the end of the chain into contact with an electrostatic generator. In both cases, after disconnecting the generator, the experimenter touched the metal nail or chain inside the flask with one hand while the other hand still surrounded the outside of the container, and the experimenter experienced an electric shock as a result.

The second discovery, in Leiden, led to the earliest commonly-used capacitor, known thereafter as the “Leyden jar.” It consisted of a metal chain to conduct charge to a sheet of metal on the inside bottom of the jar, with the outside bottom surrounded by a second sheet of metal. The Leyden jar was widely used by early experimental investigators. No longer was it necessary to connect their experimental apparatus directly to the electrostatic generator. Charge could instead be placed in the jar, and carried to the experiment.
Historical Aside

Franklin, for example, used a Leyden jar to collect charge during a thunderstorm in his famous kite experiment to show that lightning was an electrical phenomenon.

![Diagram of a modern parallel plate capacitor.](image)

**Figure 7.1:** Illustration of a modern parallel plate capacitor.

**Figure 7.2:** A schematic diagram illustrating three capacitors in parallel. The positive plates are shorted together so their potentials are equal; the same is true for the negative plates.

In both experiments the hand holding the container served as a conductor connected to a large reservoir for charge, namely the experimenter’s own body, so his hand around the container acted as one of the plates of the capacitor, his body served as the ground, and the metal inside acted as the second plate separated from the first by the glass insulator. A net charge placed on the inner conductor produced Coulomb forces acting to induce a charge of opposite sign on the hand around the container, with the excess charge free to flow into or out of the ground. Removing the container from the generator left the metal inside and the experimenter’s hand surrounding it outside with opposite net charge. But when the experimenter touched the inner conductor with his other hand, a sudden surge of charge could flow from one conductor to the other though the experimenter’s body, with shocking results.

Historical Aside

The early ideas of the Leyden jar condensing the electrical fluid (or charge) of Franklin’s single fluid theory, and of using jars whose capacities would normally be measured in pints and quarts, is the origin of the commonly used terms “capacitor” and “condenser.”
The capacitors used in modern electronics are less cumbersome than the Leyden jar, though the principle of operation is the same. In a simple parallel plate capacitor (Figure 7.1), two nearby conducting plates are separated by an insulator between them. If the two plates are connected to opposite terminals of a battery, one plate acquires a positive charge, \( Q \), and the other a negative charge, \(-Q\). The electric field between the two plates, and therefore the potential difference between them, is proportional to the charge \( Q \) producing the field.

The capacitance, \( C \), whose value depends on the detailed construction of the individual capacitor, is then defined to be the proportionality constant, \( C = Q/V \), and its value states how much charge is stored per volt of applied potential difference. By convention, we take \( Q \) in this expression to be that on the positively charged plate, so that \( C \) is always positive. The unit of capacitance is the Farad, named after Michael Faraday. The SI symbol for the farad is \( \text{F} \), and one Farad is equal to one Coulomb per Volt. It happens, however, that 1 F is a truly huge capacitance, and typical capacitors are more conveniently measured in microFarads, with \( 1 \mu\text{F} = 10^{-6} \text{F} \), sometimes even in picoFarads, with \( 1 \text{pF} = 10^{-12} \text{F} \). But because the commonly used abbreviation for “micro-Farad” originated before present conventions for the names and abbreviations of physical units were uniformly followed, the commonly encountered abbreviations “MF” and “mf” have also come to be used to mean “micro-Farad,” rather than “milli-Farad” as SI conventions would require and “mF” or “micro-micro-Farads” was common parlance for pF. Capacitors in older devices will use these markings.

In order to charge the capacitor, a battery or generator must move charge from one plate of the capacitor to the other through a potential difference thereby doing work. Thus a capacitor not only stores opposite charge on its two plates, but also stores electric energy. As we move charge through the circuit to build up the potential difference across the capacitor, the potential difference through which the charge must be moved at each instant is

\[
V(t) = \frac{Q(t)}{C}.
\]

(7.1)

For any one specific capacitor, the energy stored in it should depend only on its final charge, not on the history of how the charge was built up. Therefore, to determine the energy stored, we can assume the charge was built up at a constant rate starting from zero and make the analysis easier. Then \( V_{\text{avg}} \), the average value of the potential over this time interval, is half
the final $V$, and is therefore equal to $V_{\text{avg}} = \frac{V}{2} = \frac{Q}{2C}$. We would expect $V_{\text{avg}}$ multiplied by the total $Q$ that is passed from one plate to the other through this average potential difference to give the work that was done and the total energy stored, which is then

$$U_C = QV_{\text{avg}} = Q \cdot \frac{Q}{2C} = \frac{Q^2}{2C}.$$ (7.2)

in agreement with the result previously derived. We have used Equation (7.1) in the last step to substitute for $Q$.

The methods of integral calculus can be used to prove that the work done is indeed the charge multiplied by the time-average of the potential energy, provided the charge and therefore the potential energy difference, are built up at a constant rate.

Calculus can also show that the energy stored in a capacitor is given by Equation (7.2) regardless of the particular function of time, $V(t)$. The work $dW$ done to move a small charge $dq$ through the circuit from one plate to another with potential difference $V(q)$ between them is $dW = V \ dq$. Therefore, the work done to place a total charge $Q$ when voltage $V$ appears across the capacitor is the sum of all the small works on these increments of charge and the potential energy stored is equal to the total work done, or

$$U = \int_0^Q V(q) \ dq = \int_0^Q \frac{q}{C} \ dq = \frac{1}{C} \int_0^Q q \ dq = \left[ \frac{q^2}{2C} \right]_0^Q = \frac{Q^2}{2C} = \frac{1}{2}CV^2.$$ (7.3)

Capacitors, like resistors, can be connected either in series as shown in Figure 7.3, in parallel as shown in Figure 7.2, or in complex arrangements that can be broken down into combinations of parallel and series sets of capacitors. For capacitors in parallel (Figure 7.2), the voltage $V$ across each capacitor is the same and the total charge stored $Q_T = C_T V$ is
the sum of the charges $Q_i = C_i V$ on the positive plates, then

$$C_T = \frac{Q_T}{V} = \frac{1}{V} \sum_{i=1}^{N} Q_i = \frac{1}{V} \sum_{i=1}^{N} C_i V = \sum_{i=1}^{N} C_i. \quad (7.4)$$

Capacitors in parallel add to yield an equivalent larger capacitance like resistors in series add to form a larger equivalent resistance.

For capacitors in series, the charge on opposite plates is $\pm Q$ and charge conservation dictates the same $Q$ for each capacitor. The voltage $V_T = Q/C_T$ across all of the series is the sum of the potential differences $V_i = Q/C_i$ across each so that in Figure 7.3

$$\frac{1}{C_T} = \frac{V_T}{Q} = \frac{1}{Q} \sum_{i=1}^{N} V_i = \frac{1}{Q} \sum_{i=1}^{N} \frac{Q}{C_i} = \sum_{i=1}^{N} \frac{1}{C_i}. \quad (7.5)$$

Capacitors in series have their reciprocals add to the reciprocal of an equivalent capacitance like resistors in parallel. Thus we note an immediate difference between the math for resistors and capacitors.

While the mathematical analysis for capacitors in series and in parallel is strikingly similar to the corresponding analysis of series and parallel resistors, the physical difference between the roles of capacitors and resistors in a circuit should always be kept in mind. The capacitor responds to an applied potential difference by storing charge on its plates that it can later discharge to produce a brief current in the direction opposite to that which charged it like a spring. The capacitor stores energy in the electric field between its plates. Contrarily, a resistor responds to an applied voltage by allowing a current to flow through it and by dissipating energy and converting it to heat like the frictional force. The present experiment studies a combination of both behaviors by examining the short-lived current that a capacitor produces when it discharges through a resistor.

**Checkpoint**

Compare the capacitance of two capacitors to the “equivalent capacitance” of the same two capacitors in series. Which, if either, is bigger? Is there enough information to tell?

**Checkpoint**

When two capacitors are connected in parallel, and the capacitance of one of them is increased, how does the overall capacitance of the circuit change? Why?
Suppose a capacitor in the circuit shown in Figure 7.4(a) has charge $\pm Q$ on its plates. At time $t = 0$ the switch is moved to position 2 and electrons from the negatively charged plate become free to flow through the resistor to the positively charged plate. Charge conservation demands that the rate of charge leaving the capacitor to flow through the resistor is the charge per unit time, or the electric current $I(t) = dQ(t)/dt$, passing through the wire and the resistor. According to Ohm’s law, $I(t)$ is proportional to the voltage, $V_R(t)$, across the resistor at each instant, $t$, and is given by $I(t) = V_R(t)/R$.

When the switch is in position 2, $V_R(t) = V_C(t)$ and is also proportional to the charge still on the capacitor through Equation (7.1), $V_C(t) = Q(t)/C$. If we put all of this together, we see that

$$-R \frac{dQ}{dt} = RI(t) = V_R(t) = V_C(t) = \frac{Q(t)}{C}$$  \hspace{1cm} (7.6)

and the rate that charge is being lost from the capacitor’s plates is proportional to the instantaneous charge itself. The negative sign at the start is needed because $Q(t)$ is decreasing while the switch is in position 2. We can divide by $-R$ and use Equation (7.1) to substitute for charge to find

$$\frac{dV_C}{dt} = -\frac{V_C(t)}{RC}$$  \hspace{1cm} (7.7)

and the capacitor voltage’s rate of change is also proportional to the voltage itself.

**General Information**

In a great many different physical problems, as here, it happens that the rate of change of some physical quantity is proportional to its value at the time. Such a relation holds also, for example, in radioactive decay as well as in population growth of living cultures.

**Checkpoint**

What effect would decreasing the capacitance $C$ have on the time it takes to deplete half the stored charge and why?

**Checkpoint**

When two capacitors are connected in series, and the capacitance of one of them is decreased, how does the overall capacitance change? Why?

This is such a common situation that the mathematicians have solved it in general. In the Appendix we have detailed some of the reasoning that leads to the solution. Any time
we have a derivative of a function proportional to the function itself like
\[
\frac{df(x)}{dx} = \frac{f(x)}{\lambda}, \tag{7.8}
\]
where the constant \( \lambda \) does not depend upon \( x \), the solution to the equation is always the exponential function
\[
f(x) = f_0 e^{\frac{x}{\lambda}} \tag{7.9}
\]
where \( f_0 \) is a constant of integration and can be determined using knowledge of the initial conditions, \( f(0) = f_0 e^0 = f_0 \). In our particular case, \( x = t, f = V_C \), and \( \lambda = -RC = -\tau \). Those familiar with calculus can verify this solution by differentiating Equation (7.9) and substituting into Equation (7.8). Using these results we can immediately write the capacitor’s voltage as an exponential function of time,
\[
V_C(t) = V_0 e^{\frac{-t}{\tau}}, \tag{7.10}
\]
with the decay time constant defined by \( \tau = RC \).

**Checkpoint**

Explain, directly in terms of the physical nature of capacitance and resistance rather than from the mathematical form of the formula for the time dependence of \( Q, I, \) and \( V_C \) in the \( RC \) circuit, what effect increasing \( R \) would have on how long it takes for half the stored charge to be depleted. Why?

After a very long time, \( t/\tau \) becomes a very large number and \( e^{-t/\tau} = 1/e^{t/\tau} \approx 0 \). After a very long time the capacitor’s voltage and plate charge becomes very close to zero. Suppose we flip the switch back to position 1 while the capacitor is thus discharged. In this case \( V_C(0) = 0 \) and the battery begins to charge the capacitor through the resistor. Kirchoff’s loop rule yields
\[
0 = V_0 - V_R(t) - V_C(t) = V_0 - RI(t) - V_C(t). \tag{7.11}
\]
From Equation (7.1) we can find that
\[
I = \frac{dQ}{dt} = \frac{d}{dt} (CV_C) = C \frac{dV_C}{dt} \tag{7.12}
\]
and Equation (7.11) can be written as
\[
V_0 = RI(t) + V_C(t) = V_C(t) + RC \frac{dV_C(t)}{dt}. \tag{7.13}
\]
In the Appendix we also show that this equation can be solved, that its solution is
\[
V_C(t) = V_0 \left( 1 - e^{-\frac{t}{\tau}} \right), \tag{7.14}
\]
and that the time constant is once again $\tau = RC$. We can immediately verify the initial conditions

$$V_C(0) = V_0 \left(1 - e^{-\frac{0}{\tau}}\right) = V_0(1 - 1) = 0$$

(7.15)

are satisfied and we note that these initial conditions are automatically provided by the discharged capacitor resulting from above after a long time. We can also see that after a very long time the capacitor’s voltage is very close to $V_0$. But this is the initial conditions for the development above for a discharging capacitor. Once again we flip the switch back to position 2 and let the capacitor discharge for a long time. In our experiment we will use a square wave generator as the battery and the switch; the square wave generator will flip the switch and wait for a long time, flip the switch back and wait for a long time, flip the switch again and ... until we turn off the function generator.

We will watch the capacitor’s voltage increase and decrease as described by Equation (7.10) and Equation (7.14), respectively, using an oscilloscope and we will measure the time constant, $\tau$, for several combinations of $R$ and $C$. From the manufacturer’s measurements of $R$ and $C$, we will verify that $\tau = RC$. While keeping in mind that $\tau = RC$, we will select one resistor, $R_1$, and we will verify Equation (7.4) and Equation (7.5) with four significant digits of precision. To see how this works we will divide Equation (7.5) by $R_1$,

$$\frac{1}{\tau_s} = \frac{1}{R_1C_s} = \frac{1}{R_1} \sum_{i=1}^{N} \frac{1}{C_i} = \sum_{i=1}^{N} \frac{1}{R_1C_i} = \sum_{i=1}^{N} \frac{1}{\tau_i}. \quad (7.16)$$

Our apparatus can measure $\tau$ to four significant digits of precision if we are careful. We will use $R_1$ for all of our measurements so that only the different $C$’s can be responsible for the different $\tau$’s. If we use different $R$’s, that amounts to dividing the respective different terms in Equation (7.5) by these different $R$’s and we then cannot expect the result to be equal in that case. Algebra demands that we divide every term by the same constant, $R_1$, for the equation to remain valid.

Similarly, we can multiply Equation (7.4) by $R_1$ to find that

$$\tau_p = R_1C_p = \frac{R_1}{R_1} \sum_{i=1}^{N} C_i = \sum_{i=1}^{N} R_1C_i = \sum_{i=1}^{N} \tau_i. \quad (7.17)$$

If Equation (7.4) is true and if the rules of algebra are valid, then Equation (7.17) must also be true. Since we can measure $\tau$ to better than four significant digits, we can construct circuits containing individual capacitors and measure their time constants to four significant digits. We can use these capacitors to construct $RC$ circuits with these capacitors in series and then in parallel and we can measure their time constants to four significant digits. We can verify that the time constants obey Equation (7.16) and Equation (7.17) to four significant digits. Since the only difference between Equation (7.5) and Equation (7.16) is the multiplicative constant $R_1$, we have also verified that Equation (7.5) is valid to four significant digits. Since the only difference between Equation (7.4) and Equation (7.17) is the multiplicative constant $R_1$, we have also verified that Equation (7.4) is valid to four significant digits.
Helpful Tip

This exercise in verifying the equivalence formulas for series and parallel capacitors implements a strategy that is typical in science. It utilizes our ability to measure one quantity (the time constants) well in order to verify a physical relation well. The time constants will be different for every $R$ and the circuit might not emphasize resistance to begin with, but now that we know that the formulas work, they can be applied to any circuit containing capacitors.

Helpful Tip

We could have used the GW Instek Ohm meter to measure $R_1$’s resistance to four significant digits and then have used the measured time constants to calculate measured values for the two capacitances. These capacitances then could also be used to verify the equivalent capacitance formulas; however, that strategy would have entailed the additional uncertainty of the measured resistance and would not have demonstrated the more valuable strategy we actually utilized.

Figure 7.5: Sketch of the RC circuit on the breadboard. Be sure to short the grounding tabs together as shown by the red circles. One oscilloscope connector can plug into the side of the function generator connector as shown. This circuit has two capacitors in parallel. Replace one of the shorting jumpers to place the capacitors in series.

7.2 Apparatus

Pasco’s 850 Universal Interface for our Windows based computer can digitize voltage signals and transmit them to the computer. Pasco’s Capstone program accepts these signals and can emulate a computer-based oscilloscope to observe the time dependence of the voltage
\( V_C(t) \) across the plates of the capacitor and the square-wave generator’s output. We have a plug-in circuit breadboard that we can use to construct electrical circuits quickly. Figure 7.5 is a sketch of our circuit showing the square-wave generator that we use for a battery and switch, the series resistor and capacitor, and the oscilloscope needed to observe the square-wave and the capacitor’s voltage. We will adjust the square-wave generator so that for half of its cycle its output is \( V_0 \) and for half of its cycle its output is 0. Having the square-wave generator output be \( V_0 \) is exactly the same as having the switch in Figure 7.4 in position 1. Having the square-wave generator output be 0 is exactly the same as having the switch in Figure 7.4 in position 2. We will increase the period of oscillation (decrease the frequency) so that the capacitor has plenty of time to charge and to discharge, respectively, before the switch is flipped to begin the next sequence.

A suitable setup file for Capstone can be downloaded from the lab’s website at

http://groups.physics.northwestern.edu/lab/rc-circuits.html

The computer also is equipped with Vernier Software’s Ga3 (Graphical Analysis 3.4) program. We will copy the data containing the exponentially decaying capacitor voltage to Ga3 and we will allow Ga3 to fit our data to the decaying exponential function. The computer will choose the correct time constant to have the fitted function agree with our data as well as possible. The computer will then tell us what this time constant turns out to be and will also tell us how well the fitted function agrees with our data. If we are careful, we will see that this will allow us to determine the time constants to better than four significant decimal digits. A suitable setup file for Ga3 can also be downloaded from the website.

### 7.3 Procedure

Begin by connecting the circuit in Figure 7.5 using the plug-in breadboard. The banana jack connectors of the square-wave generator and the Pasco’s voltage sensors have a tab or bump on one side to indicate a connection to the coaxial cable’s shield. The instruments connect this shield to earth ground via the round prong of their power plugs. It is therefore necessary for proper operation that all of these bumps be connected to the shunts at the bottom of the circuit instead of being connected to the resistor. The shunt will connect them all together; since they are all connect to earth ground anyway.

Once the circuit is connected, login to the computer and start Pasco’s Capstone program. Open the lab’s “RC Circuits” webpage and execute the Capstone setup file. Press the
“Record” button at the bottom left to begin the experiment. This causes the 850 Interface to produce a pulse stream that alternates between 0 and 6 V.

**Helpful Tip**

The “Signal” button along the left edge can be used to adjust the pulse rate or the pulse height. Once the new parameters are satisfactory, pressing “Signal” again will remove the control.

The “Record” button becomes the “Stop” button (and vice versa) when it is pressed. Additionally, if you hover the mouse cursor over the graph, a toolbar will appear across the graph’s top that contains a “one shot” icon with a big red ‘1’. Clicking on this button will cause the recording to stop automatically after a single period of output signal. Clicking the button again will toggle the one-shot function off. Once the decaying response is satisfactory, enable the one-shot function to record a single capacitor discharge for further analysis.

You probably now see a square-wave and a decaying exponential. You might also note that they are jumping all over the place. It will probably be necessary to rescale the time axis by dragging the numbers. We need to establish $t = 0$ for the waveforms rather than let the computer generate it sort of at random. We do this by triggering the oscilloscope. On the left side of the oscilloscope’s toolbar is a trigger button. Click on this and the graph will then stand still. The trigger symbol will change colors to match Channel A’s data; the selected channel will have a light blue background in the top right of the Scope’s window. You can select the other channel by clicking its square at the top right of the window. Select the channel displaying the square-wave. The de-selected data will be very faint. You might note that the exponential graph is increasing rather than decreasing. We want to study the exponential decay; however, so we need to display the decreasing half of the waveform. Right-click the trigger toolbar button and select “Decreasing” instead of “Increasing”. Now you should see the correct half of the waveforms and the trigger symbol now should point downward.

### 7.3.1 Observing Exponential Decay on the Oscilloscope

Adjust the waveform to be as much like Figure 7.7(a) as you can manage and note the time constant. You can estimate the time constant by noting that Equation (7.10) predicts that

$$V_C(\tau) = V_0 e^{-\frac{\tau}{\tau}} = V_0 e^{-1} = \frac{V_0}{e} \approx \frac{V_0}{3}. \quad (7.18)$$

So, the time constant is the time between the waveform beginning to decay, $V_C(0) = V_0$, and the time when the waveform is about 1/3 of the way between the bottom of the square-wave and $V_0$. Watch this time interval and try all four combinations of $RC = R_1C_1$, $R_2C_1$, $R_2C_2$, and $R_1C_2$. For each combination record whether the time interval increased or decreased and approximately by what factor. You should try to decide for yourself whether it is possible
that $\tau = RC$ so record your observations for later review. If $R \rightarrow 2R$, does the time constant double? If $C \rightarrow C/5$, does the time constant become a fifth as large? If it is not possible that $\tau = RC$, then we obviously do not need to waste time performing a detailed experiment.

### 7.3.2 Measuring the Time Constant Using an Oscilloscope

Now, we want to pick some combination of $RC$ and to measure the time constant more accurately. It is permissible simply to keep whatever $R$ and $C$ is already installed. First, we need to adjust the function generator’s frequency so that its period is about 5-6 time constants. If the decaying waveform is like Figure 7.7(a), then we want the positive square wave edge just barely off of the display. Click the Measurement Tool button on the graph’s toolbar. Right-click on the Tool’s origin and choose Tool Properties. Increase the Significant
Figures to 5. Click OK. Drag the Tool’s origin to the top left such that the two dotted lines intersect on the exponentially decaying graph. Write the \((t, V_C)\) coordinates in your table as \(t_0\) and \(V_0\). As noted above \(V_C(\tau) = V_0/e \approx 0.3679V_0\). Multiply the value you measured as \(V_0\) by 0.3679 (or divide by \(e\)) and write the product in your Data table. Now drag the measurement tool until the capacitor voltage is as near as possible to this product. Write this new \((t, V_C)\) in your table and compute your measurement of \(\tau = t - t_0\). If your result is significantly different than \(\tau = RC\), you are doing something wrong. Once your measurement is correct, choose another \((t_1, V_1)\) and repeat the measurement. Are you convinced that every initial point, \((t_i, V_i)\), will yield the same time constant?

As one option we might repeat this process 5-10 times and perform statistics on the time constants (mean and standard deviation). The standard deviation is our best estimate for the uncertainty in each measurement of \(\tau\). And our best estimate of the time constant itself (i.e. the unknown we are trying to measure) is \(\bar{\tau} \pm s_{\bar{\tau}}\) (see Section 2.6.1).

**Checkpoint**

In the experiment, we first determine the time constant by finding the difference between a voltage \(V_0\) and \(V_0/e\). Does it matter which value of \(V_0\) we choose for doing this? Why?

Now use the manufacture’s specified values to predict this time constant

\[ \tau = RC \]

and estimate the uncertainty in this prediction. Since the only values for \(R\) and \(C\) that we have are known only to within 5%, we expect that the prediction made from these values also has uncertainty that result from these 5% tolerances. We can estimate the uncertainty in our prediction of \(\tau\) from the uncertainties in \(R\) and \(C\) using the product formula, Equation (2.7),

\[
\frac{\delta \tau_{RC}}{RC} = \sqrt{\left(\frac{\delta R}{R}\right)^2 + \left(\frac{\delta C}{C}\right)^2} = \sqrt{0.05^2 + 0.05^2} = 0.071 \quad \text{so} \quad \delta \tau_{RC} = 0.071 RC \quad (7.19)
\]

for the manufacturers’ specified values. Completely specify your prediction for this time constant like \(\tau_{RC} = (RC \pm \delta \tau_{RC})U\). Be sure not to get your predictions mixed up with your measurements.

**7.3.3 Measuring the Time Constant Using a Least Squares Fit**

Now we want to copy our data into Vernier Software’s Ga3 program for more detailed analysis. At the bottom in the center of Capstone’s area, set the sampling rate to common and about 10,000 samples/s. In the table at left, verify that both time and capacitor voltage have four significant figures. Use the two toolbar buttons above the table to adjust the displayed precision and find the point where the capacitor voltages first decreases to noise.
level (about 0.1 V). If necessary decrease the signal generator frequency to allow \( V_C \) to decay to noise level. Drag the mouse to select all time and capacitor voltage above that point, ctrl-c to copy the data, and paste into Ga3 at row 1 under Time. If data are already in Ga3, “Data/Clear All Data” before pasting to prevent old data from remaining at the bottom of the table.

Ga3 should plot the data points and automatically fit to exponential decay after only a few seconds. Verify that the model line passes through your data points and that the uncertainties are included in the fitting parameters box. Enter your names, the \( R \) and \( C \) used, etc. into the text box, and print a copy for your notebooks. Copy and paste the graph and fit for use in your report. Once you have valid copies, “Data/Clear All Data”, and go back to Capstone to measure the next experiment.

### 7.3.4 Verifying the Capacitance Formulas

**WARNING**

Change *ONLY* the capacitance between fitting experiments from this point forward. **DO NOT CHANGE the resistance!**

**Helpful Tip**

The next four measurements of time constant are quite precise. Make sure your time constant has uncertainty of a few microseconds and round the uncertainties to two significant digits. Repeat bad data. Make sure you keep exactly the right number of digits in your time constants and that you round them correctly to match your uncertainties. See Section 2.6.4.

Since \( \tau = RC \) and we can measure \( \tau \) very precisely, we basically have a capacitance meter. As long as we do not change the circuit’s resistance, \( \tau \) and \( C \) are always proportional to each other. If we had access to a *standard capacitance*, we could measure it using our fitting procedure and use this measurement to compute the proportionality constant (\( R \)) needed to make the measurement equal to the standard capacitance. This *same* \( R \) would continue to relate \( \tau \) to \( C \) until someone changed the resistance; at that point the ‘meter’ would need re-calibrated to the standard capacitance.

The fact that our meter has not been calibrated to yield *Farad* units does not mean that it is useless. Whatever units (seconds in this case) our meter produces will cancel from Equation 7.4, Equation 7.5, Equation 7.16, and Equation 7.17. Alternatively, measuring \( \tau_1 \) and \( \tau_2 \) in seconds causes Equation 7.16 and Equation 7.17 to predict \( \tau_s \) and \( \tau_p \) in the same units (seconds). We can use our measurements of \( \tau_s \) and \( \tau_p \) then to verify or to contradict those predictions.
Measure the time constants of each capacitor individually, $\tau_1$ and $\tau_2$, and the two capacitors in series, $\tau_{sM}$, and the two capacitors in parallel, $\tau_{pM}$. Try to get these done quickly so that you can verify Equation 7.16 and Equation 7.17 predicts the $\tau_p^s$ and $\tau_p^p$ that you measured. If the most significant three digits do not agree, repeat the experiment that is responsible for the bad data before continuing. If the parameters box needs more significant digits, right-click the box, “Properties...”, and increase the significant digits as needed. Too many digits can be rounded correctly; too few has already been rounded too much making the computer’s round-off the largest error in the experiment...

Correctly specify all four of your time constants in your Data section: $\tau_1$, $\tau_2$, $\tau_{sM}$, and $\tau_{pM}$, each should be specified to five or more digits in the parameters box. Round each uncertainty to two significant digits and then round the time constants to match. Make sure they have the same units and round them to the same number of decimal places. Don’t forget to record the correct units as well. If more digits are needed, right-click on the parameters box and increase the number of displayed digits appropriately.

Use Equation (7.17) to predict a value of the time constant for the parallel combination of capacitors,

$$\tau_p^p = \tau_1 + \tau_2,$$

and note that only $\tau_1$ and $\tau_2$ were used to compute this value; finding the uncertainty in this sum is straightforward using Equation (2.6),

**Helpful Tip**

Be careful to note in your notebook that this result is a prediction. We will compare this with the direct measurement above; but to do so, we will need to know which is which.

Now use Equation (7.16) to predict a value of time constant for the series combination of capacitors,

$$\frac{1}{\tau_p^s} = \frac{1}{\tau_1} + \frac{1}{\tau_2}. \quad (7.21)$$

Use at least five significant digits of $\tau_1$ and $\tau_2$ and keep at least five significant digits for $\tau_s$. Since $\tau_1$ and $\tau_2$ are used to compute $\tau_s$ and each have experimental uncertainty in them, we expect that $\tau_s$ computed from $\tau_1$ and $\tau_2$ will also have uncertainty in it due to these uncertainties.

### 7.4 Analysis

First, recall that one combination of $\tau = RC$ was first measured manually in Section 7.3.1. This same combination was then fitted to Equation (7.10) and its time constant was measured as a fitting parameter, $(\tau_1 \pm \delta \tau_1)$. Finally, this same combination was then predicted from the manufacturers’ specified component values, $(RC \pm \delta RC)$. We would like to know whether these three measurements of time constant are self-consistent.
Checkpoint

Which of these three measurements yields the best time constant? Why?

Use the best of these three time constants as a standard for comparison and compare the other two to it. Use the strategy in Section 2.9.1 to decide which of the measured time constants agree with this standard.

Statistics can tell us that the two numbers are not the same but it cannot tell us why. The answer to why is ours to figure out. At this level, the student probably made measurement and/or computational mistakes. It is also possible that our experiment was influenced by its environment and that we have not adequately allowed for or compensated these influences. It is possible that our model of our experiment and/or the physical phenomenon under study have assumptions that are not realized. Finally, it is possible that our theory is wrong; just look what happened when we tried to use Ohm’s law on an LED!

Discuss subtle experimental errors that have not been included in the $\delta\tau$’s above. If possible, estimate how each might have affected your measurements. Might some of these be large enough to explain any observed disagreement?

7.4.1 Notes

When your analysis is complete your notebook should address the following:

- Do your data exponentially decay? How do you know?
- Does our differential equation solution predict the correct behavior and time constant?
- Are Equation (7.5) and Equation (7.4) valid for series and parallel capacitors? How well were these equations tested?

7.5 Conclusions

What equations are supported by your data? Define all symbols and communicate with complete sentences. (If you have numbered your equations in your report, you may reference them here instead of repeating them.) How well have you ‘proved’ each equation? Have you measured anything that is likely to be of use in the future? How might you improve the experiment? How might you apply your observation(s)?
Helpful Tip

Review your graphs and your Data section generally to remind yourself what physics we have used in our experiment. If these physics tools led to good agreement, then the tools are supported.

Even if some physics was tested before, we might have used it in a different way or on a different component than before. These new demonstrations are suitable for your conclusions.
7.6 Appendix: The exponential decay law

We use calculus to solve the original differential equation,

\[
\frac{dV_C}{dt} = -\frac{V_C(t)}{\tau}. \tag{7.22}
\]

We use algebra to rearrange the equation to the form

\[
\frac{dV_C}{V_C} = -\frac{dt}{\tau}. \tag{7.23}
\]

and integrate both sides,

\[
\ln(V_C(t)) - \ln(V_C(0)) = -\frac{t - 0}{\tau}
\]

\[
V_C(t) = V_C(0)e^{-\frac{t}{\tau}} = V_0 e^{-\frac{t}{\tau}}. \tag{7.24}
\]

From this we can find the charge on the positive capacitor plate and the current in the circuit.

\[
Q(t) = \frac{V_C(t)}{C} = \frac{V_0}{C} e^{-\frac{t}{\tau}} = Q_0 e^{-\frac{t}{\tau}}. \tag{7.25}
\]

and

\[
I(t) = \frac{dQ(t)}{dt} = -\frac{Q_0}{\tau} e^{-\frac{t}{\tau}} = -I_0 e^{-\frac{t}{\tau}}. \tag{7.26}
\]

When the capacitor is charging, Kirchhoff’s loop rule gives

\[
0 = V_0 - V_R(t) - V_C(t) = V_0 - RI(t) - V_C(t) = V_0 - RC \frac{dV_C}{dt} - V_C(t) \tag{7.27}
\]

or

\[
V_0 = \tau \frac{dV_C}{dt} + V_C(t). \tag{7.28}
\]

This inhomogeneous differential equation is harder to solve, but the solution always contains the solution of the homogeneous differential equation that we solved above. The rest of the solution is any particular solution to the inhomogeneous differential equation;

\[
V_C(t) = V_0 \tag{7.29}
\]

is such a particular solution as we can see by substituting it into Equation (7.27), so

\[
V_C(t) = V_0 + A e^{-\frac{t}{\tau}} \tag{7.30}
\]
is the solution we seek for some value of $A$. We use the fact that at the moment the switch is flipped the capacitor is completely discharged,

$$0 = V_C(0) = V_0 + Ae^0,$$

to find that $A = -V_0$ and our solution is

$$V_C(t) = V_0 - V_0 e^{-\frac{t}{\tau}} = V_0 \left(1 - e^{-\frac{t}{\tau}}\right). \quad (7.31)$$

From this we can also find the charge on the positive plate and the current flowing in the circuit to be

$$Q(t) = \frac{V_C(t)}{C} = \frac{V_0}{C} \left(1 - e^{-\frac{t}{\tau}}\right) = Q_0 \left(1 - e^{-\frac{t}{\tau}}\right) \quad (7.32)$$

and

$$I(t) = \frac{dQ(t)}{dt} = \frac{Q_0}{\tau} e^{-\frac{t}{\tau}} = I_0 e^{-\frac{t}{\tau}}. \quad (7.33)$$

We note that the current in the circuit is momentary when charging and when discharging the capacitor and the sign is opposite in the two cases as it should be.
Chapter 8

Experiment 6: Magnetic Force on a Current Carrying Wire

8.1 Introduction

Maricourt (1269) is credited with some of the original work in magnetism. He identified the magnetic force centers of permanent magnets and designated them as North and South poles, referring to the tendency of the North pole to seek the north geographic pole of the earth. A generalization of the interaction between these so called force centers can be summarized as: opposite poles attract and similar poles repel. Gilbert (1600) recognized that the earth itself was a natural magnet, with magnetic poles located near the geographic poles, only the north geographic pole is a South magnetic pole.

As it was originally conceived, the magnetic field, $\mathbf{B}$, designated the direction and magnitude of the force (per pole strength) on a North pole of a permanent magnet. This was directly analogous to the electric field, $\mathbf{E}$, which was defined as having the direction and magnitude of the electric force (per unit charge) on a positive test charge. The magnetic field emanates from the North pole of a magnet and appears to end on the South pole, but now we believe that $\mathbf{B}$ always forms closed loops.

Ampere determined that a current carrying wire is affected by a magnetic field. Both magnitude and direction of the magnetic force is described by the cross product,

$$\Delta F_m = I \Delta L \times \mathbf{B}, \quad (8.1)$$

where the direction of $\Delta L$ is in the direction of flow of positive current, $I$. The direction is ascribed to the spatial variable $\Delta L$ rather than the current, $I$, for the convenience of integration. If we consider infinitesimal lengths the expression can be written without loss
in generality as
\[ dF_m = I \, dL \times B. \] (8.2)

Figure 8.1 shows the vector relationship between the magnetic force, \( dF_m \), the magnetic field, \( B \), and current direction, \( L \).

**Checkpoint**

If a current is moving along a wire aligned with a magnetic field, what is the magnitude and direction of the force of magnetism on the wire?

![Image of a conductor, permanent magnets, and a non-uniform magnetic field]

**Figure 8.2:** A sketch of the conductor, the permanent magnets, and the non-uniform magnetic field. For lines through the center, the field is horizontal in our apparatus so that the force can be downward.

In this experiment we will use high strength Neodymium permanent magnets to create an intense magnetic field in a localized region of space. We will place a current carrying conductor in this region, measure the resulting magnetic force, and compare our experimental value with the force predicted by the force equation.

The magnetic field created in the pole gap will be relatively uniform within the gap but will drop off over a finite region just outside the gap. It is necessary to know approximately where the field drops off, to define where along the wire the magnetic field exists, and thus contributes to the magnetic force.

![Graph of magnetic field intensity versus position of the Hall probe]

**Figure 8.3:** An example graph of the magnetic field intensity versus the position of the Hall probe.
8.2 Magnetic Field in Magnet Gap

If one had detailed information about the strength of the field as a function of position through the gap, one could do a calculation of the integral

\[ F_m = I \int_0^L dx \times B(x). \]  

(8.3)

Figure 8.2 shows a diagram of the typical magnetic field produced in the gap between two strong Neodymium magnets. A profile of the magnetic field through the gap is shown in Figure 8.3. You will obtain an approximate profile of your specific gap using a Hall probe to measure the field every half centimeter along a line through the center of the gap. For finite segments of distance through the gap we can approximate Equation 8.2 with

\[ \Delta F_i = I \Delta L_i \times B_i \]  

(8.4)

where \( \Delta L_i \) is the unit of length of resolution used to map the magnetic field. A good approximation to the total force can be estimated by summing the components over the range of measured \( B \) field values,

\[ F = \sum_{i=1}^{N} \Delta F_i = \sum_{i=1}^{N} I \Delta L_i \times B_i = I \Delta L \times \sum_{i=1}^{N} B_i, \]  

(8.5)

since the same values of \( \Delta L \) and \( I \) apply to each segment.

**Checkpoint**

- What are the units of magnetic force? What are the units of magnetic field?

**Checkpoint**

- How is the direction of the magnetic force oriented with respect to the directions of magnetic field and current which produced it?

8.3 Apparatus and Procedure

This experiment will be performed in two parts. First, we will use a Hall effect magnetic field sensor to measure the magnetic field profile, \( B(x) \), between the neodymium (Nd) ceramic magnets. See Section 8.3.1. Second, we will place a current-carrying wire between these same magnets and use a digital mass scale to measure the magnetic force for various charge flow rates (currents). See Section 8.3.3. The apparatus overview is shown in Figure 8.4.
Chapter 8: Experiment 6

Figure 8.4: A photograph of the apparatus used to measure magnetic field $B(x)$, electric current $I$, and magnetic force $F_m$. Relevant components are indicated.

Hopefully, the Theory above clarifies how these two pieces fit together.

8.3.1 Measuring the B field profile

Two neodymium permanent magnets are placed in the gap of a soft iron yoke to create a very intense magnetic field in the gap between the magnets. The strength and extent of this field will be mapped using a Hall effect probe. First, slide the base of the Hall probe stand until it touches the meter stick guide as shown in Figures 8.5 and 8.6. Keep the base against the meter stick and verify that it passes precisely through the center of the gap between the magnets. If necessary, nudge the magnet yoke as shown in Figure 8.5 to position the center of the gap symmetrically about the probe. Be careful henceforth not to disturb the magnet again until the experiment is complete.

**WARNING**

The Hall probe has been very carefully adjusted to pass through the center of the magnetic field. It has also been oriented to be most sensitive to the field so do not adjust the Hall probe. Handle the base of the stand gently. If you alter the careful alignment, you can expect to get bad data.
Slowly and carefully slide the Hall probe through the gap between the magnets to observe the strength of the magnetic field.

**Helpful Tip**

The post holding the magnets and yoke will rotate in the hole in the table. It can be rotated back... When the Hall probe passes cleanly midway between the two magnets while the Hall probe stand is rubbing the meter stick, the magnets are in the right place. Each time the magnets are disturbed, this is the way to put them back. See Figure 8.5.

You should see the field strength increase and then decrease as shown in Figure 8.3. The maximum field should be 550-700 mT if the Hall probe’s alignment is still optimized. Otherwise, tell your lab TA so he(she) can arrange to have it aligned again.

Now we want to measure the profile, $B$ vs $x$, of your magnetic field more carefully. We will use Vernier Software’s Graphical Analysis 3.4 (Ga3) to analyze our data. A suitable setup file can be downloaded from the lab’s website at

http://groups.physics.northwestern.edu/lab/magnetic-force.html

Choose page ‘1:Profile’ for the first part of the experiment.

Carefully slide the Hall probe to the left side of the magnets and find a position where the field strength is about 50 mT. Set the position to be the nearest half-centimeter to minimize entry errors. Review Figure 8.6 and record the magnetic field and the position of one edge of the probe stand on the meter stick in Ga3. Record other observations in your notebook. Move the probe and stand a half-centimeter along the rule toward the magnets and record the magnetic field and position again. Repeat the operation through the gap until the probe has scanned the entire gap and the field is again about 50 mT. When you finish, your graph should be similar to Figure 8.3. You can also construct a table of $x$ and $B$ in your notebook. Estimate the uncertainties $\delta x$ and $\delta B$ and enter these in Ga3 as indicated also.
Figure 8.6: A photograph of the apparatus configured to measure the magnetic field profile $B(x)$. The peak field strength must be at least 550 mT.

Helpful Tip

Watch for data points that look “out of place”. Your eyes have evolved to notice subtle patterns and departures from patterns, so use the tools you have. Re-measure these bad data points carefully and enter the correct values.

Evaluate the term $\Delta x \sum B_i$ by summing the column of $B$ values in your table and multiplying by the interval, $\Delta x$, you used while making the measurement. Alternately, let Ga3 perform the integral (sum) of the area under the curve to obtain $\text{Area} = A_I = \Delta x \sum B_i$. Draw a box around the data points and “Analyze/Integrate” or click the integrate button on the toolbar.

Checkpoint

What is the Hall probe used to measure in this experiment? If $\Delta x = 1.00 \text{ cm}$ instead of 0.50 cm, why would the calculated area not be twice as big?

Checkpoint

How should we orient the Hall probe to measure the magnetic field most accurately?
(Option)

Use the considerations in the appendix to propagate $\delta x$ and $\delta B$ through to find $\delta A_I$.

### 8.3.2 (Option) The Average Magnetic Field

Calculate the average (or effective) magnetic field for your magnet using your area, $A_I$, and your full width at half maximum, $W$,

$$B_{\text{avg}} = \frac{A_I}{W}. \quad (8.6)$$

This value characterizes your magnet in case you want to use the magnet in another experiment. We now simply need to measure $W$ and to multiply by this average field instead of measuring all of those fields and integrating. We also need the error in our average field. Can you guess how to calculate this? It will involve $\delta B$ and $\delta x$.

### 8.3.3 Measuring the Magnetic Force

First, carefully remove the Hall effect $\mathbf{B}$ field probe and set it aside. (See Figure 8.4.) Next, we need to determine the direction of the magnetic field vector with a magnetic compass. See Figure 8.7. Since the compass magnets can be reversed in the strong field of the Nd magnets, it is first necessary to determine the north pole (N) of the compass magnet. We use earth’s magnetic field to do this as shown in Figure 8.7(a). Hold the compass far from the Nd magnets and thump the handle to randomize any friction. The north pole will point somewhat northward but mostly downward. The blue end in Figure 8.7(a) is the compass’ north pole and will be attracted to the south pole of other magnets. If I were performing the experiment, I would add an entry in my notebook about the color of the compass’ north pole.

#### Historical Aside

The standard convention for magnets is that the north pole is red. Evidently, the magnet in the illustration has had its magnetization reversed at least once.

#### WARNING

The Nd magnets are strong enough to pull the compass’ gimbal apart. Hold the handle firmly and bring the compass close enough to orient the compass’ magnet but not close enough for a strong force. See Figures 8.7(a) and (b).
The magnet is attached to a support bar and oriented as shown in Figure 8.4 and in greater detail in Figure 8.7. Now, use the compass to determine which side of the gap has a north pole and which side has a south pole. Hopefully, the two poles are different so that the field does not cancel in the middle. Refer to Figure 8.7 and be careful not to get the compass too close.
CHAPTER 8: EXPERIMENT 6

Helpful Tip

It is also a good idea to make permanent notes about which end of the compass is attracted to which side of the gap... Such notable facts make great notes. As you figure out which direction $B$ points, make that observation permanent as well. These details are not needed in the final report; however, they make arriving at a correct report more probable.

An electronic mass scale whose pan has been fitted with a non-ferromagnetic, non-conducting wire support is carefully placed atop a lab jack such that the conductor placed atop the wire support is as shown in Figure 8.4. The conductor and balanced are raised into position such that the wire’s height is the same as the magnetic field’s center; the conductor should now pass through exactly the same points whose magnetic fields we measured earlier—at least as precisely as we can arrange it in the real world. The conductor should also be oriented perpendicular to the magnetic field to maximize the force (recall the vector cross-product).

WARNING

The lab jack has been carefully adjusted so that your wire passes through the center of the magnetic field and locked in place. This adjustment is specific to your lab station, so do not move the lab jack to another station. With effort, you can still defeat the jack lock and get bad data... Instead of adjusting the lab jack, refer to Figure 8.8.

Checkpoint

To get the best data, how should we orient the conductor while measuring the magnetic force? What path should the current pass through?

WARNING

The mass scale is very sensitive; but it is also very fragile. Any force more than a few Newtons applied to the supports or the weighing pan will break the scale’s sensor.

Slide the lab jack, mass scale, brass wire, and support between you and the yoke. See Figure 8.8. Temporarily lift the brass wire off its supports and lower it between the yoke and the wire supports. Carefully slide the lab jack away from you so that the yoke and magnets are between the supports. Follow the green path in Figure 8.8(a) with the center of the wire as you move the lab jack. It might be necessary to remove objects from between the lab jack and the meter stick. This will allow you to insert the wire between the magnets.
CHAPTER 8: EXPERIMENT 6

Figure 8.8: Three photographs illustrating how we can insert the current wire and force sensor into the magnetic field. Position the scale as shown in (a), lift the wire, and move the wire toward the magnets and downward so the wire will pass under the magnet yoke as indicated by the green path. Simultaneously, slide the scale toward the magnets taking care that the yoke passes between the two black wire supports (b). Continue sliding the scale until the wire can rest on the supports between the magnets (c).

and to raise it above them so that the supports are on opposite sides of the gap between the magnets. A soft touch is required to avoid moving the magnets and yoke; if necessary, remove the scale and use the Hall effect probe to align the yoke as depicted in Figure 8.5. At this point you can then replace the wire on the supports and have the wire pass through the maximum field between the magnets. Carefully adjust the jack’s position to center the wire between the magnets and align the wire with the front edge of the meter stick. Since the meter stick guided the field probe, this will increase the probability that the current flows along the same path that we profiled earlier. (See Section 8.3.1.)

The mass scale will be used to measure the magnetic force on the conductor as it conducts a current in the magnetic field. These force measurements will have units of grams (g) because we usually want to know the mass of the object placed on the scale and not its
weight \( (w = mg) \). The scale determines these values of mass by measuring the gravitational force (weight) that the mass exerts on the scale. Note that it is the force that the scale senses not the mass. If the scale were in space, it would always read zero regardless of the mass attached to its pan. In this experiment we are tricking the scale by exerting a force that is not a weight; but the scale continues to divide this force by \( g = 9.807 \text{ m/s}^2 \) and to display the result in ‘grams’. The scale cannot know that the force we are exerting is a magnetic force; this does not change the fact that to get the magnetic force we need to convert the scale’s ‘mass’ reading to kg and to reverse the scale’s division by \( g \).

![Figure 8.9: A photograph of the current source’s and the mass scale’s displays detailing relevant controls and indications. The current “Hi-Lo” scale switch should be in the “Hi” position and the top current scale should be read. The mass scale photograph shows the decimal point and the gram unit indicator.](image)

Use the right-hand-rule, your observed magnetic field direction, and the direction current flows to predict the direction of the magnetic force. Note that conventional current flows out of the red power supply terminal and into the black terminal. Keep in mind that a previous group might have reversed the leads in the power supply and, if this is the case, the wire colors are backwards.

Set the power supply to 3.0 Amps and switch it off. See Figure 8.9. Tare the balance to read zero grams. You will need to support the scale while pressing the ‘tare’ button; using one hand to squeeze the button and the scale’s bottom usually works best. Make sure the wires are not in contact with the scale’s sample pan, be careful not to disturb the current wires, switch the power supply back on, and note the effect of 3.0 Amperes on the mass scale. If the balance reading is not positive, your current is flowing in the wrong direction. Verify that the force of magnetism is in the correct direction as predicted by the cross product in Equation (8.1); use the right-hand-rule. Record this vector analysis in your notebook and, perhaps, your Data. To reverse the current, it is far quicker and safer just to reverse the leads.
in the power supply jacks. Read the effective mass as indicated by the scale, and multiply
this number by the acceleration of gravity. This is the magnetic force due to the current of
3.0 Amps passing through the conductor in the magnetic field. Take several readings of this
force for different currents in 0.5 Amp increments. After setting each current, switch off the
power supply and tare the balance for zero current before carefully switching the power back
on and reading the force. Estimate uncertainties for your current measurements and your
force measurements.

**Helpful Tip**

As you read the mass scale, keep in mind that there are two digits after the decimal
point. See Figure 8.9(a). Also, verify that ‘g’ units are selected.

Make a plot of the force verses the current. Select the ‘2:Force’ page from the left end
of the toolbar in Ga3. Fill in the ‘mass’ and current values into the table and watch the
computer plot your F vs I. Verify that the force is directly proportional to the current. Do
your data points lie in a straight line? Ga3 can compute force from effective mass; just enter
the measured current and scale readings (in grams). Verify that the force column has units
in Newtons (N) and has calculation formula

\[ 9.807 \times \text{“mass”} \div 1000 \]

in the ‘Equation’ edit control. Estimate \( \delta m \) and \( \delta I \) and enter these into the columns provided.

**Checkpoint**

How is the magnitude of the magnetic force related to the magnitude of the current
carried in a wire? How is the magnitude of the magnetic force related to the magnitude
of the magnetic field? How is the magnetic force experimentally related to other
measurable forces? Why did we divide \( mg \) by 1000?

Be sure your graph shows force (not mass) vertically and current horizontally. Drag a
box around your data points so that all rows in your table turn grey. “Analyze/Automatic
Fit.../Proportional” and “Try Fit”. This will choose the best ‘A’ to represent your data such
that \( F = AI \). Make sure that the solid fit curve passes through your data points and OK.
Drag the parameters box off of your data. If the uncertainty in \( A \) is not shown, right-click
the parameters box and select “Fit Options.../Show Uncertainties”. This dialog box also
controls the significant figures in the fitted parameters.

Once you are satisfied with your force vs. current graph, remove the wire and mass scale
by reversing the process depicted in Figure 8.8. Your lab performance evaluation should
suffer if you leave your station in a mess.
8.4 Analysis

Do your data indicate that your measured magnetic force is directly proportional to the current? Is this consistent with Equations (8.5) and (8.3)? Use the strategy in Section 2.9.1 to compare the slope of your force graph from Section 8.3.3 to the area under your magnetic field graph from Section 8.3.1. Can you think of any reason they should be the same? Are your units the same? Consider the uncertainty ($\delta A$) in your slope; does your slope and uncertainty bracket your area? Can you think of any errors that we have overlooked?

What does statistics say about how well your two measurements (slope vs. area) agree? Can you think of any subtle sources of error that we have not included? List as many as you can and briefly predict how each might affect your measurements. If we had figured out how to add these to our comparison, they would have made the tolerance range larger. Might any of these additional sources of error be large enough to explain why your Difference is so large?

8.5 Conclusions

What equation have we verified? Preferably reference a labeled equation in your report; otherwise, define all symbols and communicate with complete sentences. What quantities have you measured that we might need again? Don’t forget to include your units and uncertainty. If these values are prominently featured in a table, you might reference the table instead. How might we improve the experiment? Can you think of applications for any of your observations or apparatus?

Helpful Tip

Review Appendix E from time to time to refresh your memory about report contents.
8.6 Appendix: The Uncertainty in Profile Integral

The uncertainty in magnetic field strength, $\delta B$, moves the graph of $B(x)$ up or down. We can form an area of uncertainty by multiplying the uncertainty in $B$ by ‘an appropriate range’ of $x$. See Figure 8.3. The field extends to infinity; it simply gets weaker and weaker with distance. But this also indicates that we know the field to a smaller and smaller range as distance increases. Many students have guessed at this point that the ‘appropriate’ distance needed has to do with the full width at half maximum (FWHM) $W$.

Our magnets do not generate a field that is easily modeled mathematically; however those that do suggest that $2W < D < 3W$ is the distance we need for this purpose. For convenience we will use

$$\delta_1 = 2W\delta B.$$  

The uncertainty in Hall probe position, $\delta x$, moves the left half (and the right half) of the ‘bell’ curve left or right. This forms an area of uncertainty if we multiply by the height of the bell

$$\delta_2 = B_{\text{max}}\delta x.$$  

Each of these independent uncertainties are reduced by the fact that the area is formed by all of the data points and thus the computed area averages the uncertainties among the several points. To decide how much averaging has occurred, it is necessary to divide by ‘an appropriate number’ of points. I have probably spoiled the surprise at this point, but this will be determined by the full width at half maximum and the distance between points. We will use

$$N = \frac{2W}{\Delta x}.$$  

Altogether,

$$\delta A_1 = \sqrt{\frac{\delta_1^2 + 2\delta_2^2}{N}} = \sqrt{\left(\frac{2W\delta B}{2W}\right)^2 + 2\left(B_{\text{max}}\delta x\right)^2} \frac{\Delta x}{2W} = \sqrt{\frac{2\left(W\delta B\right)^2 + \left(B_{\text{max}}\delta x\right)^2}{W}} \frac{\Delta x}{W}.$$  

(8.7)
Chapter 9

Experiment 7:  
Electromagnetic Oscillations

9.1 Introduction

The goal of this lab is to examine electromagnetic oscillations in an alternating current (AC)  
circuit of the kind shown in Figure 9.1. The circuit differs from that in the $RC$ circuit  
experiment by including the solenoid coil of wire, labeled “$L$.” When the switch is moved  
from position 1 to position 2, the charged capacitor discharges; however, the voltage across  
the capacitor does not now simply decay to zero as we saw for the $RC$ circuit. Since an  
inductor stores energy in its magnetic field, this circuit’s capacitor voltage oscillates between  
positive and negative values as time passes; we saw similar behavior when we studied the  
pendulum. Electrical resistance dissipates the capacitor’s energy ($P = V^2/R$) so that its  
voltage slowly goes to zero ($U_C = CV^2/2$). Since even good wires and oxidized metal in  
contacts have some resistance, the amplitude of the oscillations seen in this $RLC$ circuit will  
also slowly go to zero.

![Figure 9.1: A schematic diagram of our RLC circuit. The resistance is not explicitly added but some of our other components have resistance. The battery and switch will be replaced by a square-wave generator.](image)

![Figure 9.2: A plot of an exponentially decaying oscillation. Each time the switch flips, the RLC circuit responds with damped oscillations.](image)

In fact, there is a strong analogy between mechanical oscillations (the pendulum) and
electromagnetic oscillations. The same differential equation of motion describes both; only
the constants in the equation \( \frac{1}{l} \leftrightarrow L, \ g \leftrightarrow C, \ b \leftrightarrow R \) and the changing function
\( \theta(t) \leftrightarrow V_C(t) \) are different. These physical characteristics of the mechanical oscillator
therefore correspond to their specific electromagnetic counterparts in the AC circuit studied
here. The same equations that describe the oscillation of mechanical quantities also describe
the oscillation of electromagnetic quantities and many results apply equally for mechanical
oscillators and electromagnetic oscillators. To explain the oscillatory behavior of the circuit
in Figure 9.1 and the close analogy between mechanical and electromagnetic oscillations, we
first discuss Faraday’s law, inductors, and alternating current circuits.

**Historical Aside**

Other oscillators include a mass on a spring that leads to tuning forks, bells, and guitar
strings. An electron in an atom in the process of emitting a photon is an oscillator.
Atoms in solids oscillate about discrete lattice points. Because these oscillators all act
alike, we once utilized “analog computers” to simulate these other phenomena using
electric circuits at a much lower cost.

Every electric current generates a magnetic field. The magnetic field around the
moving charge can be visualized in terms of

**Checkpoint**

What physical quantity in an AC cir-
cuit plays the same role that frictional
forces play in a mechanical oscillator?
Why?

magnetic lines similar to the electric lines
of force for the electric field. At each point
the direction of the straight line tangent to a
magnetic field line gives the direction of the
magnetic field in complete analogy to the
electric lines of force. Magnetic field lines
are not lines of force since, as we saw last
week, magnetic forces are perpendicular to
the fields not parallel to them.

The solenoid used as an inductor in this
experiment (component \( L \) of Figure 9.1)
consists of a long insulated wire wound
around a cylinder with a constant number of

**Figure 9.3:** A sketch of the magnetic field
lines formed by current flowing in a solenoidal
coil of wire. As shown, the field is formed
by current flowing upward on the side of the
coil toward us and downward on the side away
from us. The right-hand-rule predicts the
north pole on the left.
turns per unit length of the cylinder. The current, \( I \), through the wire produces a magnetic field as illustrated in Figure 9.3. To examine the role of the solenoid, consider more generally any area, \( A \), enclosed by a closed path \( P \) in space, as illustrated in Figure 9.4. Then with the magnetic field at each point in the region we can define the flux

\[
\Phi = \int_{\text{area}} \mathbf{B} \cdot \mathbf{da} \quad (9.1)
\]
as essentially the number of lines of the \( \mathbf{B} \) field (or the “amount” of \( \mathbf{B} \) field) passing through the area. If the lines of the magnetic field are perpendicular to the area, \( A \), then the flux \( \Phi \) is simply the product of the magnetic field times the area. The dimensions of the magnetic flux are Webers (or Wb) with

\[ 1 \text{ Wb} = 1 \text{ Volt-second}. \]

Michael Faraday (in England) and Joseph Henry (in the United States) independently discovered that a change in flux induces an electromotive force (emf). Henry made the discovery earlier but published it later than Faraday. Specifically, a changing flux through the area enclosed by the path in Figure 9.4 produces an emf around the closed path. The emf is defined as the work per unit test charge that the electric field would do on a small positive charge moved around the path, \( P \), and is equal in magnitude to the rate of change of the flux through the circuit,

\[
\mathcal{E} = -\frac{d\Phi}{dt}. \quad (9.2)
\]

The physical relation given by Equation (9.2) is usually referred to in physics as \textit{Faraday’s law}. The minus sign is required by Lenz’s law, and indicates that the induced emf is always in a direction that would produce a current whose magnetic field opposes the change in flux (or tends to keep the flux constant).

Suppose the loop defining the area, \( A \), is made of copper wire with ohmic resistance \( R \). Ohm’s law means that any emf induced in the loop will cause a current \( I = \mathcal{E}/R \) to flow through the loop. A coil of \( N \) turns with the same changing flux through each would be equivalent to \( N \) single loops in series, each with the same emf thereby producing the emf,

\[
\mathcal{E} = -N \frac{d\Phi}{dt}. \quad (9.3)
\]

As the capacitor shown in Figure 9.1 discharges, the current would simply decrease exponent-
tially if the solenoid had no effect. But the increasing current changes the flux through the solenoid and thereby induces an emf acting on the solenoid itself (referred to as a “back emf”) that slows the rate that the current increases. Once the capacitor has discharged completely, the inductor has built up a substantial current and a substantial magnetic flux. This current will begin to charge up the capacitor with the opposite (negative) polarity and this negative emf will now begin to decrease the inductor’s current. As the capacitor’s charge (and emf) increases in the negative direction, the rate of current change also becomes more negative until it finally becomes zero once again. Now the magnetic flux is zero, but the capacitor has a substantial negative charge and voltage. This negative emf will, similarly, cause a negative current and flux to increase in absolute value until the capacitor has expended its electric energy while building magnetic energy. This magnetic energy will once again charge the capacitor with positive voltage completing the cycle. This positive capacitor voltage will then begin a new and similar cycle that will repeat endlessly unless the energy is dissipated instead of being swapped back and forth between electric energy and magnetic energy.

For a pendulum we found that gravitational potential energy \((U_g = mgL(1 - \cos \theta))\) slowly became kinetic energy in the pendulum bob \((K = mv^2/2)\) until \(\theta = 0\) and \(U_g = 0\). At this point the bob had substantial momentum that forced the pendulum past the equilibrium \(\theta < 0\) and \(U_g > 0\) until it had built up substantial gravitational potential energy. This gravitational potential energy and kinetic energy was swapped back and forth until the friction in the pivot and air resistance dissipated the energy. When we studied the pendulum, we included no mechanism to dissipate the energy so the prediction was that the oscillations would be persistent; including a damping term \(\tau \omega = -b \frac{d\theta}{dt}\) in the pendulum’s torques would have made that motion equivalent to this.

To examine the circuit more quantitatively, note that for a coil of \(N\) turns of wire and each with flux, \(\Phi\), the total flux, \(N\Phi\), through the coil must be proportional to the current through the wire,

\[
N\Phi = LI
\]  

(9.4)

The proportionality constant in this relation is the self-inductance \(L\) given by

\[
L = N \frac{\Phi}{I}
\]  

(9.5)

The inductance is a characteristic property of the inductor determined by its geometry (its shape, size, number of windings, material near the windings, and arrangement of windings), just as the capacitance of a capacitor depends on the geometry of its plates and on whatever separates them from each other. The units of \(L\) are Volt-second/Ampere with

\[
1 \text{ V·s/} \text{A}=1 \text{ Henry}
\]

(with the symbol for Henry being H not to be confused with Hertz, Hz).
CHAPTER 9: EXPERIMENT 7

Checkpoint

What is magnetic flux? What is Faraday’s law? Who first discovered it?

Because Equation (9.5) relates the flux through the circuit to the current at each instant of time, the time rate of change of the two sides of Equation (9.5) must also be equal. Since the left side of the equation is the total flux linkage, \( N\Phi \), its rate of change is the induced emf, while the rate of change of the right hand side is \( L \) times the rate at which the current changes at each instant. Therefore the back emf is equal to \( L \) multiplied by the rate of change of the current; equivalently, we simply differentiate both sides of Equation (9.4),

\[
\mathcal{E} = -N\frac{d\Phi}{dt} = -L\frac{dI}{dt}.
\]  

(9.6)

Helpful Tip

Note that we use \( \mathcal{E} \) for the emf induced by changing flux in the coil. When analyzing a circuit containing an inductor, we need \( V_L \) across the inductor that causes the flux to change. These action-reaction effects have opposite signs.

Checkpoint

What physical features determine the inductance of a solenoid?

9.1.1 Energy Considerations

As a charge, \( dQ \), flows through the circuit, it gains energy \( V \, dQ \), where \( V = L \, dI/dt \). Thus, the energy lost by the charge (which is the energy given to the inductor) is

\[
dU_L = V \, dQ = \left(L \frac{dI}{dt}\right) \, dQ = L \, dI \left(\frac{dQ}{dt}\right) = LI \, dI.
\]  

(9.7)

If we start with zero current, and build up to a current \( I_0 \), the energy stored in the inductor is

\[
U_L = \int_{U_L(0)}^{U_L(I)} \, dU = L \int_0^I I' \, dI' = L \left[\frac{I'^2}{2}\right]_0^I = \frac{1}{2}LI^2
\]  

(9.8)

This energy is stored in the form of the magnetic field. When the switch, \( S \), in Figure 9.1 is in position 1, the capacitor becomes charged. Eventually, the capacitor has the full voltage \( V_0 \) across it and has energy \( CV_0^2/2 \) stored in the electric field between its plates. Once
we set the switch to position 2 at $t = 0$, current flows through the inductor building up a magnetic field. As the voltage oscillates, the energy oscillates between being magnetic energy of the inductor and electric field energy of the capacitor. Some of the energy is lost to the environment as heat because of the ohmic resistance of the circuit and any other conductors that get linked by the flux lines of the inductor. These losses decrease the maximum of the energy $CV^2/2$ stored in the capacitor (and $LI^2/2$ stored in the inductor) in each successive cycle. The amplitude as measured by the maximum of the voltage, $V_C$, across the capacitor therefore decreases from each cycle to the next.

### Checkpoint

In terms of energy, when can a system oscillate?

### Checkpoint

What is the energy in an inductor in terms of quantities such as charge, current, voltage?

### 9.2 Damped Oscillations in RLC Circuits

We next need to examine the precise time dependence of the voltage $V_C(t)$ and of the current $I(t)$ in a circuit such as that illustrated in Figure 9.1. Although the circuit includes a capacitor, an inductor, and a resistor, the behavior of the circuit is determined by the total resistance $R = R_L + R_{\text{misc}}$, total inductance $L$, and total capacitance $C$ of the entire circuit, rather than the value added explicitly by the components. Each component of the circuit contributes to these three physical aspects of the circuit; the resistor, for example, also has a slight capacitance and inductance like the inductor has resistance in its wire and capacitance between its windings. Since the resistance of the short wires is fairly negligible, the resistance measured from one side of the capacitor to the other in Figure 9.1 is seen to be $R = R_L + R_{\text{misc}}$. The relation between voltage and current for these three elements are summarized in Table 9.1, together with the SI symbol and the expressions for the energy associated with each physical quantity. In any closed circuit the sum of the voltages across the components must be zero, so that

$$V_L + V_R + V_C = 0. \quad (9.9)$$

Based on the expressions for the voltage differences in Table 9.1,

$$L \frac{dI}{dt} + IR + V_C = 0; \quad (9.10)$$
but the current is due to charge flowing onto (or off of) the capacitor plates, so
\[ I = \frac{dQ}{dt} = C \frac{dV_C}{dt}, \quad \frac{dI}{dt} = C' \frac{d^2V_C}{dt^2}, \quad (9.11) \]
and
\[ LC \frac{d^2V_C}{dt^2} + RC \frac{dV_C}{dt} + V_C(t) = 0 \]
\[ \frac{d^2V_C}{dt^2} + \frac{R}{L} \frac{dV_C}{dt} + \frac{V_C(t)}{LC} = 0. \quad (9.12) \]
This equation occurs so often in math and physics that the mathematicians have solved it in general. We can simply look it up in books containing tables of integrals and solutions to differential equations. To make finding this particular solution easier, differential equations are classified. This one has a second derivative so it is second order. The terms having the function, \( V_C \), in them are multiplied by constants and not functions of time, so its class has constant coefficients. Since the function on the right is 0, this class is homogeneous. We need to look for solutions of homogeneous second order differential equations with constant coefficients. We will find that
\[ \frac{d^2f}{dt^2} + 2\gamma \frac{df}{dt} + \omega_0^2 f(t) = 0 \quad (9.13) \]
has the solution
\[ f(t) = f_0 e^{-\gamma t} \cos(\omega' t + \varphi) \quad (9.14) \]
where \( \omega' = \sqrt{\omega_0^2 - \gamma^2} \) and \( f_0 \) and \( \varphi \), are constants of integration that can be found from initial conditions. If we compare Equation (9.12) to Equation (9.13), we can identify
\[ 2\gamma = \frac{R}{L}, \quad \omega_0^2 = \frac{1}{LC}, \quad \text{and} \quad f(t) = V_C(t) \quad (9.15) \]
so
\[ \gamma = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad \omega' = \sqrt{\omega_0^2 - \gamma^2}, \quad \text{and} \quad V_C(t) = V_0 e^{-\gamma t} \cos(\omega' t + \varphi). \] (9.16)

You can convince yourself that Equation (9.14) gives the correct solution by substituting into Equation (9.12).

The function \( V_C(t) \) is that shown in Figure 9.2 and Figure 9.5. It is the product of an oscillatory term and an exponential decay function that damps the amplitude of the oscillation as the time increases. The period of the cosine (and sine) function is \( 2\pi \) radians so we can find the period, \( T \), of the oscillations using
\[ 2\pi = \omega' T \quad \text{and} \quad T = \frac{2\pi}{\omega'}. \] (9.17)

The damping factor, \( \gamma \), does not affect the period, \( T \), of the oscillatory term very much. The transfer of energy back and forth from the capacitor to the inductor is illustrated in Figure 9.5. In this figure the current, \( I = \frac{dQ}{dt} \), is also plotted.

At the time marked 1 in Figure 9.5 all the energy is in the electric field of the fully charged capacitor. A quarter cycle later at 2, the capacitor is discharged and nearly all this energy is found in the magnetic field of the coil. As the oscillation continues, the circuit resistance converts electromagnetic energy into thermal energy and the amplitude decreases.

\[ \text{Figure 9.5: Sketches of capacitor voltage and circuit current showing how electric energy and magnetic energy swap back and forth between the capacitor and the inductor.} \]

**Checkpoint**

Which of these circuits – \( RC, RL, LC, RLC \) – can produce oscillations? Explain.
9.2.1 Observing damped oscillations in a \( RLC \) circuit

We connect a capacitor, a resistor, and an inductor in series on a plug-in breadboard as shown in Figure 9.6. The set-up differs from the schematic diagram in Figure 9.1 by including connections to the computer-based oscilloscope to monitor the time dependence of the input and capacitor voltages. Also, just as in the case of the RC circuit experiment, the switch in Figure 9.1 is replaced by the square-wave output of the function generator. If you have forgotten how to use the computer-based oscilloscope, it would be a good idea to read the section of the fifth lab write up in which its operation is described.

### Table 9.2: A list of the units of the three main quantities characterizing the circuit.

<table>
<thead>
<tr>
<th>Name</th>
<th>symbol</th>
<th>SI Unit Name</th>
<th>Unit</th>
<th>Unit Definition</th>
<th>Short</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistance</td>
<td>( R )</td>
<td>Ohm</td>
<td>( \Omega )</td>
<td>Volts per Ampere</td>
<td>V/A</td>
</tr>
<tr>
<td>Inductance</td>
<td>( L )</td>
<td>Henry</td>
<td>( H )</td>
<td>Volt-second per Ampere</td>
<td>Vs/A</td>
</tr>
<tr>
<td>Capacitance</td>
<td>( C )</td>
<td>Farad</td>
<td>( F )</td>
<td>Coulomb per Volt</td>
<td>C/V</td>
</tr>
</tbody>
</table>

9.3 Experimental set-up

Download the Capstone setup file from the lab’s website at

http://groups.physics.northwestern.edu/lab/em-oscillations1.html

Insert the components into the breadboard of your setup as shown in Figure 9.6. We will measure a 0.1\( \mu \)F capacitor and a 1.0\( \mu \)F capacitor, but it is not important which we observe first.

We will also need to install the inductor, the function generator, and two channels of our oscilloscope.

Observe the shape of the input pulses through Channel A of the computer-based oscilloscope display, while simultaneously observing the capacitor’s voltage, \( V_C \), with Channel B. Power on Pasco’s 850 Universal (computer) Interface and start their Capstone program. Click the “Record” button at the bottom left. You should see a decaying oscillation quite similar to Figure 9.2. If you do not, check your circuit connections or ask your TA for assistance.

Click the “Signal” button at the left and play with the “Frequency” of the pulse train. You should see that higher frequencies don’t allow the oscillations time to decay completely before
they are re-energized. The waves are sketched in Figure 9.7. Is the beginning point constant or does it jump around? We want to use a frequency low enough that the measurement noise is almost as large as the oscillations before the pulser changes state; this will cause a repeatable waveform that is the same every time.

9.4 Procedure

Sketch the displayed \( V_C(t) \) in your Data; some artistic skill will be useful from time to time. Be sure to note which capacitor you are using. What are the tolerances and units? Indicate on this sketch, using small squares as markers, the points where the energy of the system is all in the electric field of the capacitor. Indicate with a circle where it is all in the form of magnetic energy. Draw a legend that defines what these symbols mean. Use the oscilloscope to measure the period of oscillation, \( T \). See if you can remember how to use the Measuring Tool. It may be more accurate to measure the time of \( n \) periods, rather than just one. Calculate the angular frequency \( \omega' = \frac{2\pi}{T} \).

Hover the mouse cursor over the graph so that a toolbar appears at the top of the graph display. Click the Selector Tool (an icon with a blue square surrounded by eight small grey squares) and adjust the selected area so that only one decaying sinusoid is yellow. Scroll the table until you find the beginning or the end of the yellow entries; these are the table entries that correspond to the yellow data points on the graph. Drag with the mouse to select only these yellow entries and ctrl+c to copy these times and capacitor voltages to Windows’ clipboard.

9.4.1 Observe the ‘Motion’

Run Vernier Software’s Graphical Analysis 3.4 (Ga3) program. We have also provided a suitable setup file for Ga3 on the website. Click row 1 under Time and ctrl+v to paste your Pasco data into Ga3. After a few seconds, Ga3 should plot your data points and automatically fit them to

\[
V_c \exp(-g*t) \cos(w*t+p)
\]
Compare this expression to Equation (9.16). Since Ga3 cannot accept Greek symbols, it was necessary for us to use English letters: \( \gamma \leftrightarrow g \), \( \omega' \leftrightarrow w \), and \( \varphi \leftrightarrow p \). Be sure the solid fit model curve passes through all of your data points; the computer is not smart enough to do this for you. When manually fitting, keep in mind that \( V_{ac} \) is the initial amplitude (half of the peak-to-peak voltage), \( g \) is the rate that the amplitude decays, \( w \) is the oscillation frequency, and \( p \) is a horizontal phase offset. If the curve goes too quickly to 0, reduce \( g \); in fact you can use \( g = 0 \) until you get the oscillation adjusted if you like. To get an approximate \( w \), just type in a number (1,2,3...) and add zeros until the oscillations get too fast; now that you have the correct order of magnitude you can fine tune from most significant to least significant digit until the frequency matches that of your data. Choose \( -\pi < p < \pi \) since this is the period of sine and cosine; align the zeros with your data’s zeros. Finally, adjust \( g \) to get the correct amplitude envelop time dependence. Once the fit model matches your data points pretty well, click ‘OK’. Now “Analyze/Curve Fit...” again and “Try Fit” again to let the autofitter try again beginning with your manual fit parameters. Once your fit is satisfactory, type your names, which capacitor was measured, and any error information you have on the data points into the Text Window. “File/Print/Whole Screen” and enter the number of people in your group into ‘Copies’.

Now swap capacitors and repeat the recording, copying, and fitting procedure for the new oscillations. You will probably want to “Data/Clear Data” in Ga3 before pasting in the new data in case the new data set is smaller than the old one. In this case several data points at the bottom will not be replaced by the new data and they will appear erroneously on your graph and analysis.

9.4.2 Predict the Inductance

Now, use these fitting parameters to determine your circuit parameters. Since there are three unknowns, \( R \), \( L \), and \( C \), but only two equations, it is necessary for us to assume a value for one of our circuit elements. If we measure the inductor’s dimensions carefully, our value of \( L \) will have the tightest tolerance of 1%. While measuring the solenoid, keep in mind that it is the wire that results in inductance; the plastic coil form adds nothing. It is therefore necessary to determine the dimensions of the wire coil. The wire was wrapped around the plastic spool so the inner diameter of the wire is the outer diameter of the round plastic tube. The flat plastic ends of the spool holds the ends of the wire coil flat so the length or height of the wire solenoid is the same as the distance between the inside surfaces of these flat spool ends. The outer diameter of the wire solenoid can be measured by looking through the transparent plastic spool ends. The number of turns of wire in the coil is written on the coil’s label along with other useful information. Record your measurements, their uncertainties, and their units.

Execute the “Inductance” applet from the website. Type your measurements into their respective edit controls and click “Calculate”. It is necessary that your measurements be entered with millimeter (mm) units, so you might have to convert your units. Additionally, the program reports the inductance with milli-Henry (mH) units instead of Henrys.
9.4.3 Measure $R$ and $L$

Note the manufacturer’s specified capacitance, $C$; both capacitors have 5% tolerance. This specified capacitance, $\gamma = g$, and $\omega' = w$ can now be used to find a predicted value for your inductance and resistance equivalent of total energy losses. Rearrange Equations (9.16) to see that

$$L = \frac{1}{(\omega'^2 + \gamma^2) C} \approx \frac{1}{\omega'^2 C}$$

(9.18)

can be used to predict $L$ and

$$R = 2\gamma L$$

(9.19)

can be used to predict $R$. Since the fitting parameter uncertainties are so much smaller than 5%, the uncertainties in these measurements is rather close to 5% as well: $\delta L \approx 0.05 L$ and $\delta R \approx 0.05 R$.

Observe both capacitors and measure $R$ and $L$ twice—once for each capacitor. Be careful that you do not mix up your predictions and measurements. Did you remember to record the manufacturer’s measurements, units, and tolerances for these components? What components have substantial resistance that the current must flow through? Do you get the same inductance and resistance for both capacitors? Discuss this with your classmates to see if there is a correlation that might be a clue to the explanation. Try to find all of the missing pieces.

9.4.4 The Infinite Solenoid

If time is short, skip this part. Determining the inductance of a finite length solenoidal coil is quite difficult, but we have used Ampere’s law to determine the inductance of an infinitely long solenoidal coil to be

$$L = \frac{\mu_0 A N^2}{h}.$$  

(9.20)

Pretend that this applies to your coil and substitute your dimensions into this formula to predict $L$. We won’t expect to get the correct answer, but we should get within a factor of 2 or 3 if the infinite solenoid formula is correct.

9.5 Analysis

Consider your measured $R$’s and $L$’s; did you get the same answers for both capacitors? Use the strategy in Section 2.9.1 to decide whether each prediction and specification are equal. Do you see a pattern?

What other subtle sources of error have we omitted from our comparisons? Might some of these be large enough to help explain our disagreement(s)? Since $L$ and $C$ store energy and return it to the circuit later, only $R$ dissipates energy. Every loss will increase $R$; can you think of any ways energy was lost from the circuit? Does your large or small capacitor
circuit have larger losses? This might be a clue you can use to locate some of the losses. Repeat this analysis for the inductor keeping in mind that the units are now H instead of Ω.

Discuss how well our solution fits our data? Is it likely that our fit would be this good if our theory were substantially incorrect? Consider the complexity of our model when answering this. Did we get the right answers for $L$? How about $R$? Is the same energy initially stored for both capacitors?

### 9.6 Conclusions

What equations have we demonstrated? References to equations in your main report are preferred. Communicate using complete sentences and define all symbols. Have we made any measurements that we might need in the future? If so note them, their uncertainties, and their units. What is the significance of the initial voltage and phase? How might our readers obtain them from their own data? What does this experiment indicate about the way we analyze circuits? What does our data say about the inductance of an infinitely long solenoid? How might we improve the experiment? Can you think of any applications for what you have observed?
Chapter 10

Experiment 8: Electromagnetic Resonance

10.1 Introduction

If a pendulum whose angular frequency is $\omega_0$ for swinging freely is subjected to a small applied force $F(t) = F_0 \cos \omega t$ oscillating with a different angular frequency, $\omega$, the force acts during part of each cycle in a direction opposite to velocity of the pendulum, thereby partially negating its effect in accelerating the pendulum during the rest of the cycle. The strongest response to the applied force would be expected when $\omega \approx \omega_0$, since then the force stays in phase with the velocity of the pendulum and can act throughout the entire cycle in the same direction that the pendulum is moving. If frictional effects are small, a pendulum starting from rest can attain a large amplitude of oscillation from the cumulative effect of even a weak oscillating force acting over many periods of oscillation, provided the applied driving force is precisely tuned to the characteristic frequency of oscillation. Far more important for practical uses than the selective response of a pendulum to the frequency of an applied force, however, is the corresponding effect in an electric circuit.

We examine this kind of behavior for electromagnetic oscillations in the RLC circuit, where the resistance, inductance, and capacitance are analogous, respectively, to friction, inertia (or mass), and a spring-like restoring force in mechanics. In the previous experiment, the square-wave generator repeatedly produced a steady voltage to charge the capacitor, followed by an abrupt change to zero; this allowed the circuit to oscillate freely at its natural frequency while the circuit resistance damped the motion to zero. These were observations of the circuit’s transient response. Since the natural circuit response is sinusoidal, it is reasonable to wonder how the circuit will respond to a sinusoidal

![Figure 10.1: A schematic diagram of our RLC circuit. This particular circuit has the four elements in series so all of the currents are the same.](image)
stimulus. It turns out that when the stimulus is first applied or when the sinusoid is quickly changed, the circuit will always respond with its transient response until they naturally damp themselves out. Once the transients are gone, the remaining motion is the steady state response. This class of differential equation is not capable of any other kinds of solutions. The most general response of RLC circuits is

\[ V_C(t) = V_{\text{transient}}(t) + V_{\text{steadystate}}(t) \]  \hspace{1cm} (10.1)

In the present experiment, we connect another RLC circuit to a sine-wave generator as shown in Figure 10.1 and observe its response for different frequencies,

\[ V(t) = V_0 \sin \omega t \]  \hspace{1cm} (10.2)

This signal has amplitude, \(V_0\), and angular frequency, \(\omega\). The circuit is called a “driven” RLC oscillator. We will see that the response will also be sinusoidal, so this is also a driven, damped harmonic oscillator. We seek to determine expressions for the resulting current, \(I(t)\), in such a circuit in what follows. We use Kirchoff’s loop rule around the circuit,

\[ 0 = V(t) - V_L - V_C - V_R \]  \hspace{1cm} (10.3)

Since \(Q = CV_C\) for a capacitor, its current is

\[ I = \frac{dQ}{dt} = C \frac{dV_C}{dt}. \]  \hspace{1cm} (10.4)

The same current flows through all of the components so this makes

\[ V_R = IR = RC \frac{dV_C}{dt} \]  \hspace{1cm} (10.5)

where we have applied Ohm’s law to the resistor and

\[ V_L = L \frac{dI}{dt} = LC \frac{d^2V_C}{dt^2} \]  \hspace{1cm} (10.6)

for the inductor. Thus we substitute from Equation (10.2) for \(V(t)\) and rearrange Equation (10.3) to find that

\[ LC \frac{d^2V_C}{dt^2} + RC \frac{dV_C}{dt} + V_C(t) = V_0 \sin \omega t \]  \hspace{1cm} (10.7)

is the differential equation that we must solve to predict the circuit’s response. This inhomogeneous second order differential equation with constant coefficients is much harder to solve than the homogeneous one from last week; however, the mathematicians have managed it. This differential equation in standard form can be written

\[ \frac{1}{\omega_0^2} \frac{d^2f}{dt^2} + \frac{1}{Q\omega_0} \frac{df}{dt} + f(t) = f_0 \sin \omega t \]  \hspace{1cm} (10.8)

128
Chapter 10: Experiment 8

Figure 10.2: Plots of amplitude (top) and phase shift (bottom) for frequencies near resonance \( f_0 \). In both cases three quality factors (3, 10, and 30) have been plotted.

where this quality factor, \( Q \), is not the capacitor (or any other) charge. In the Appendix we demonstrate a strategy to use ‘phasors’ in the solution of these complicated equations. Last week we defined the ‘resonance frequency’ of the \( RLC \) circuit to be

\[
f_0 = \frac{1}{2\pi \sqrt{LC}} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}}. \tag{10.9}
\]

This week we also will need the quality factor at resonance

\[
Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}. \tag{10.10}
\]

If \( Q \) is small, the circuit responds to a very wide range of frequencies; if \( Q \) is large, the circuit responds to a very narrow range of frequencies. This property of resonance leads us to the concept of “bandwidth”,

\[
\Delta \omega = \frac{\omega_0}{Q} \quad \text{or} \quad Q = \frac{f_0}{\Delta f}. \tag{10.11}
\]

The \( LRC \) circuit is one example of a “filter”. Signals with \( \omega_0 - \frac{\Delta \omega}{2} \leq \omega \leq \omega_0 + \frac{\Delta \omega}{2} \) are passed to the output but other frequencies are blocked. \( \Delta \omega \) is the width of the “passband” or the bandwidth of the filter. Strictly speaking this bandwidth is a little ‘fuzzy’; frequencies barely inside the passband are attenuated noticeably and frequencies barely outside the passband are passed somewhat. The edges of the passband are the frequencies where the passed power is half of its maximum.

It turns out that we can manufacture variable inductors and capacitors whose values change as we rotate a knob. By using one of these in our filter, we can vary the resonance frequency along a certain range. Increasing \( L \) also increases the quality factor and decreases the bandwidth. Increasing \( C \) decreases the resonance frequency but does not change the bandwidth.

**Checkpoint**

Can you think of a use for a circuit whose resonance frequency can be varied over the frequency range \( 88.1 \text{MHz} \leq f \leq 108.1 \text{MHz} \)?

Equation (10.10) gives the quality factor only at resonance but we also define a capacitive
and inductive $Q$ at all frequencies,

$$Q_C = \frac{1}{\omega RC} \quad \text{and} \quad Q_L = \frac{\omega L}{R}.$$  \hspace{1cm} (10.12)

The quality factor at resonance of macroscopic mechanical systems is limited to about 100 due to the prevalence of friction. The quality factors of electromagnetic systems can be $10^4$ to $10^6$. The quality factor of atomic systems can be $10^{10}$. Recently, the quality factor of a coherent laser with a cooled coherent atomic gas amplifying medium reached a new record of $10^{17}$. Resonances with large $Q$ can be used to make very precise and accurate clocks.

**Checkpoint**

What characterizes a resonance?

**Checkpoint**

What is meant by the “resonance frequency” of an $RLC$ circuit? For a swinging simple pendulum, what is the resonance frequency?

Since the power delivered to a resistor is $P = I^2 R$ and the band edge pass half of the power at resonance, we must have

$$I(\omega_\pm) = \frac{I_0}{\sqrt{2}}$$  \hspace{1cm} (10.13)

at each of the passband edges ($\omega_-$ is the lower edge and $\omega_+$ is the upper edge).

**Checkpoint**

What is the meaning of the $Q$ value of an oscillator? How does the $Q$ value change with increased resistance in the circuit? What feature of the response curve is described by the $Q$ value of the resonance?

At frequencies below resonance, the current peaks before the applied voltage. At frequencies above resonance, the current peaks after the applied voltage. Exactly at the passband edges the phase has shifted by $\varphi = \pm 45^\circ$.

We say the current “leads” the voltage for frequencies below resonance. We say the current “lags” the voltage for frequencies above resonance. The two particular cases having frequencies at the band edges are illustrated in Figure 10.3. As the applied frequency extends toward 0 Hz, the current wave at the top of the figure will move toward $\pi/2$ or $90^\circ$ phase shift. As the applied frequency extends toward higher frequencies (infinity) the wave at
CHAPTER 10: EXPERIMENT 8

The bottom of the figure will move toward $-\pi/2$ or $-90^\circ$ phase shift. This frequency dependence of phase shift is also illustrated in the bottom of Figure 10.2. In the series circuit that we have here, the largest voltage will develop across the component with the largest reactance; therefore, the largest reactance will determine the phase shift. Since

$$X_L = \omega L \quad \text{and} \quad X_C = \frac{1}{\omega C}, \quad (10.14)$$

the inductor will have the largest reactance at high frequencies and the capacitor will have the largest reactance at low frequencies. To help keep track of the phase relation for the different components, remember “ELI the ICE man”. When the inductor’s phase relation is desired, ELI reminds us that for $L$, $E$ is before $I$ where the integral of electric field is voltage; voltage leads current. When the capacitor’s phase relation is desired, ICE reminds us that $I$ is before $E$; current leads voltage. With regards to the series $RLC$ circuit, $X_C$ is larger below resonance so ICE reminds us that current leads voltage; $X_L$ is larger above resonance so ELI reminds us that voltage leads current or current lags voltage.

**Checkpoint**

What does “the current leads the voltage” mean? Give an example of when this occurs.

### 10.2 The Experiment

In this experiment we will observe the frequency dependence of the amplitude and phase shift of the current in a series $RLC$ experiment near resonance. Our inductor, capacitors, and function generator will be the same as last week, but this week we will install a $47 \, \Omega$ resistor to allow us to monitor the current on an oscilloscope (a voltage sensing device). Since Ohm’s law gives us $I = \frac{V_R}{R}$, we can divide the resistor’s voltage by $47 \, \Omega$ to get the current to be the same function of time as the applied voltage. Since there are no derivatives in this expression, the phase of the resistor’s voltage and current are equal; only the magnitudes are different and these scale as shown by the resistance itself. We will use Pasco’s Voltage Sensors and 850 Universal Interface for the personal computer and their Capstone program to emulate an oscilloscope.
10.3 Procedure

Wire up the circuit as shown in Figure 10.4. Verify that this implementation is indeed represented schematically by Figure 10.1. Be sure to short the black ground leads of the voltage sensors and generator together as shown so that one or more of them do not short out part of your circuit. Download the Pasco Capstone configuration file from the lab web site,

http://groups.physics.northwestern.edu/lab/em-oscillations2.html,

and execute it. You can probably just click the link and then allow Capstone to load it directly. Turn on the 850 Universal Interface by lightly holding the power button for a second. You can click the “Signal Generator” button displayed at Capstone’s left to display and to remove the signal generator controls. Turn on the signal generator and set the desired frequency; begin with 125 Hz or so. Click the “Monitor” button to see the stimulus and the response; the button changes to “Stop” so that clicking it again pauses the data acquisition and display.

Now watch the resistor voltage (current) on Channel B as you slowly increase the frequency of the function generator by 10 Hz increments. Continue until you notice a change in the waveforms; this should occur at frequency between 150-600 Hz. Note your observations in your Data. Watch the amplitude and the relative phase of the current. When you locate the resonance, change the increment to 1 Hz and observe the resonance in more detail. Measure the resonance frequency when the current is in phase with the applied voltage. You can change the increments to 0.1 Hz if necessary. Also, note your measurement uncertainty and units in your Data. Now we would like to measure the bandwidth.

Stop the scope at the bottom left. Select the current channel and get a Measurement Tool. Set the origin at the waveform’s valley or at the waveform’s peak. Record the resonance frequency and maximum response amplitude (either peak or peak-to-peak) in your

Figure 10.4: A sketch of the plug-in breadboard holding the RLC resonance circuit. Be sure to short the ground tabs together as shown in the red circles. The connector for Channel A can be plugged into the side of the function generator’s connector as shown.
Data. What are the units and error for this measurement? Note that the signal generator’s amplitude is 6 V; the peak-to-peak value is 12 V.

**Helpful Tip**

Measuring the peak-to-peak voltage doubles your signal and prevents bias in your data from amplifier input offsets. Measuring the peak value requires only one measurement instead of two but probably will introduce an error source you should mention in your writeup. The choice is yours. If you choose peak-to-peak measurements, you can enable the Delta Tool.

Now we want to estimate the circuit’s ‘bandwidth.’ Divide the response amplitude, \( V_0 \), at resonance measured above by \( \sqrt{2} \) as suggested by Equation 10.38. Note this value for use in the next two steps. Find the frequencies, \( V(f_+) = \frac{V_0}{\sqrt{2}} \) and \( V(f_-) = \frac{V_0}{\sqrt{2}} \), above and below resonance that each gives a resistor voltage \( \frac{V_0}{\sqrt{2}} \pm 5\% \). Note that simultaneously the phase shifts are \( \pm 45^\circ \). Record both of these frequencies and the response amplitudes (peak or peak-to-peak as measured above). Subtract these two frequencies to estimate the bandwidth, \( \Delta f = f_+ - f_- \). Record your work in your Data. Use the measured resonance frequency and bandwidth to estimate the circuit’s quality factor as in Equation 10.36. You can use the three points you have measured above to begin your observation of the amplitude’s response function below.

### 10.3.1 Resonance Amplitude Response

Now fill out Table 10.1 and type the results into Vernier Software’s Ga3 graphical analysis program. You can also download a suitable Ga3 setup file,

http://groups.physics.northwestern.edu/lab/em-oscillations2.html.

Unfortunately, your browser will open this text file itself unless you save it to your data folder first. Double-click the “RLC Resonance.ga3” that you downloaded and type in your data from Table 10.1.

**Helpful Tip**

Regardless where you save RLC Resonance.ga3 (preferable in Documents\Students), you can execute it from Mozilla’s Firefox® browser by clicking the down-pointing blue arrow at the top right; click the file from the drop-down box. It might be wise to make yourself a folder and to save your data there as you go.

Fit your data to the Lorentzian line shape in Equation (10.35). Drag a box around your data and select the “Resonance Peak” function from the bottom of the list. You may verify
Table 10.1: A place to record observations of frequency versus generator and resistor voltage.

<table>
<thead>
<tr>
<th>Formula</th>
<th>Frequency (Hz)</th>
<th>$V_{\text{gen}}$ (V)</th>
<th>$V_r$ (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0 - 3\Delta f$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 - \frac{\Delta f}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 - \frac{\Delta f}{8}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 + \frac{\Delta f}{8}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 + \frac{\Delta f}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0 + 3\Delta f$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

the model to be

$$V_r / \sqrt{1 + (Q \times (x/f_0 - f_0/x)^2)}.$$

Drag the fit parameters box off of your data. If your fit parameters box does not include uncertainties in the fitting parameters, right-click the box, “Properties…”, and select the “Show Uncertainties” checkbox.

Helpful Tip

You can copy your data table and your graph directly to your Word processor report. Click on the object to copy, “Edit/Copy”, activate Word, and ctrl+v. While the object is selected in Word, “Insert/Caption...” to give it a label. You can also adjust the object’s properties to yield a nice layout in the document.

10.3.2 Measuring $R$, $L$, and $C$

The capacitances are specified by the manufacturer to 5% tolerance. Thus we can use Equation (10.9) and the measured resonance frequency, $f_0$, to determine the inductance

$$L = \frac{1}{(2\pi f_0)^2 C}$$
Table 10.2: A place to record observation of frequency versus the time shift needed to find phase shift.

<table>
<thead>
<tr>
<th>Formula</th>
<th>Frequency (Hz)</th>
<th>$t_\delta$ (s)</th>
<th>$T (s)=1/f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0-3\Delta f$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0-\frac{\Delta f}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0-\frac{\Delta f}{8}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0+\frac{\Delta f}{8}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0+\frac{\Delta f}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_0+3\Delta f$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where we have used $\omega = 2\pi f$

Next we can use this inductance, the quality factor, and Equation (10.10) to determine the total resistance

$$R = \frac{2\pi f_0 L}{Q}.$$  

Assuming the quality factor and resonance frequency have uncertainties much less than 5%, $L$ and $R$ each will also have errors about 5% — $\delta L = 0.05 L$ and $\delta R = 0.05 R$.

10.3.3 (option) Resonance Phase Response

IF you have time, use the Smart Tools to measure the phase shift versus frequency curve. Complete Table 10.2 and enter it into Ga3. Measure the time between where the generator voltage crosses zero and where the test resistor’s voltage crosses zero. If the resistor voltage comes first (is to the left) give the time a positive sign and if it comes after the generator voltage give it a negative sign. Now verify that the phase shift column computes its values using

$$\text{“Tdiff” * “Frequency” * 6.2832.}$$  \hspace{1cm} (10.15)

Be sure your column names and units are correct. Paying attention to the phase versus frequency graph, drag a box around your data points, and “Analyze/Curve Fit…” to
Equation (10.32). “Resonance Phase” at the bottom of the equation list uses

\[
\text{atan}(Q \cdot ((f_0^2 - x^2) / (f_0 \cdot x)))
\]

as its model.

Be sure the fit model passes through your data points before you OK and drag your fit parameters box off of your data. Are your \( f_0 \) and \( Q \) the same as those you got from the other graph? The resonance frequency is much more accurate from this graph, so feedback systems are usually designed to lock to this characteristic of resonance. “File/Print...” and enter the number in your group into Copies to get graph and table printouts for each of your group’s notebooks. Also, copy and paste your table and graph into your Word document.

10.4 Analysis

Does the resonance response describe your data adequately? Part of this answer rests on the quality of the model fit(s) in your graph(s). Do the model parameters agree with the circuit parameters? Use the strategy in Section 2.9.1 to decide whether the resonance model has predicted the correct resistance and inductance.

What other subtle sources of error can you find in this experiment? Does the model result from any assumptions that might not be met? What affects magnetic flux through the inductor that we have ignored? Can you identify any likely energy losses other than circuit resistance? Are any of these other errors large enough to explain any disagreement(s)? How well does the Lorentzian distribution predicted by our circuit analysis fit our data? How about the phase shift data?

10.5 Conclusions

What equations are supported by our experiment? (If you numbered your equations, you can simply quote the references here.) Communicate using complete sentences and define all symbols. Did we measure anything worth reporting in our Conclusions? If so give their values, units, and errors. How might we improve the experiment? Have you noted potential applications for what you have observed?
10.6 Appendix: Phasors

Sinusoidal waveforms consist of amplitude, frequency, and constant phase, \( A \sin(\omega t + \phi) \). We might consider using Euler’s formula,

\[
e^{i\theta} = \cos \theta + i \sin \theta,
\]

(10.17)
to express sinusoidal waveforms using exponential notation. First, change the sign of \( \theta \) in Equation (10.17) to find the complex conjugate relation,

\[
e^{-i\theta} = \cos \theta - i \sin \theta.
\]

(10.18)

We will use “\(^*\)” to represent complex conjugation. Next, form the difference between Equation (10.17) and Equation (10.18),

\[
\frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{\cos \theta + i \sin \theta - (\cos \theta - i \sin \theta)}{2i} = \sin \theta.
\]

(10.19)

Since the terms on the left are a complex conjugate pair, we can always derive one from the other by complex conjugation (\( i^* \rightarrow -i \)); we might as well just replace \( \sin \theta \rightarrow e^{i\theta} \) with the understanding that subtracting the complex conjugate (c.c.) and dividing by \( 2i \) will return the sinusoid. We can then express the sinusoid above as

\[
A \sin(\omega t + \phi) \rightarrow Ae^{i(\omega t + \phi)} = Ae^{i\phi}e^{i\omega t} = ae^{i\omega t}
\]

(10.20)

where we have absorbed the real amplitude and constant phase into a complex phasor, \( a = Ae^{i\phi} \). Those familiar with trigonometric identities will immediately see how much easier it is to deal with the sum of two angles using exponential notation. In fact any complex number can be expressed as a phasor and this makes phasors valuable for representing impedance and current response in electronic circuits. (In the strictest sense, ‘phasor’ implies a relationship to a sinusoid having amplitude, \( |a| = A \), and phase, \( \arg(a) = \phi \).)

Although we do not yet know the capacitor’s voltage as a function of time, we can observe it on an oscilloscope to be a sinusoid and to have the same frequency, \( \omega \), as \( V(t) \). Since any sinusoid is an amplitude, a frequency, and a phase, knowing the capacitor’s response frequency allows us to write down an expression for the voltage

\[
V_C(t) = v_C e^{i\omega t}.
\]

(10.21)

This also allows us to write an expression for the circuit’s current

\[
I(t) = C \frac{dV_C}{dt} = i \omega C v_C e^{i\omega t} = i e^{i\omega t}
\]

(10.22)

where so far \( i \) and \( v_C \) are unknown phasors. We must be very careful not to confuse the current’s phasor \( i \) with the imaginary unit \( i \). Electrical engineers have developed the convention of using ‘j’ for the imaginary unit to reduce the risk of mixing them up.

137
Equation (10.22) does tell us that we must have

\[ i = i\omega C v_C. \]  

\(^{(10.23)}\)

### 10.6.1 Solution to the Motion

Now we are ready to put these phasors to work in order to solve Equation (10.8). I reproduce it here for your convenience

\[ V_0 \sin \omega t = LC \frac{d^2 V_C}{dt^2} + RC \frac{dV_C}{dt} + V_C(t). \]

First, we substitute Equation (10.21) into this to find

\[
\frac{dV_C}{dt} = i\omega v_C e^{i\omega t}
\]

\[
\frac{d^2V_C}{dt^2} = (i\omega)^2 v_C e^{i\omega t} = -\omega^2 v_C e^{i\omega t}
\]

\[ V_0 e^{i\omega t} = -\omega^2 LC v_C e^{i\omega t} + i\omega RC v_C e^{i\omega t} + v_C e^{i\omega t} \]

\[ V_0 = -\omega^2 LC v_C + i\omega RC v_C + v_C \]

where we have canceled the common factor \(e^{i\omega t}\). Next, we use Equation (10.23) to substitute for \(v_C\) and get

\[ V_0 = -\omega^2 LC \left( \frac{i}{i\omega C} + i\omega RC \frac{i}{i\omega C} + \frac{i}{i\omega C} \right) = \left( R + i \left( \omega L - \frac{1}{\omega C} \right) \right)i = zi \]

\(^{(10.24)}\)

where we have defined the “complex impedance” \(z = R + i \left( \omega L - \frac{1}{\omega C} \right)\).

Although it isn’t obvious from this development, \(V_0 = V_0 e^{i0} = v_0\) is also a complex phasor; it simply happens to have zero phase. The above development works exactly the same, however, if the applied stimulus has nonzero phase. Observant students will have noticed the similarity between Equation (10.24) and Ohm’s ‘law’. In fact we call Equation (10.24) “Ohm’s Law for AC circuits”. Observant students might also have noticed that resistors simply contribute \(z_R = R\) to the impedance and inductors and capacitors contribute

\[ z_L = i\omega L \quad \text{and} \quad z_C = \frac{1}{i\omega C}; \]

\(^{(10.25)}\)

this is why we define the reactances of these components by Equations (10.14).

We can represent the impedance of Figure 10.1 using phasor notation,

\[ Ze^{i\varphi} = z = R + i \left( \omega L - \frac{1}{\omega C} \right) \]

\(^{(10.26)}\)
where the magnitude is

\[ Z = |z| = \sqrt{z^*z} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \]  \hspace{1cm} (10.27)

and phase satisfies

\[ \varphi_z = \arg(z) = \tan^{-1}\left(\frac{\Im(z)}{\Re(z)}\right) = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right). \]  \hspace{1cm} (10.28)

The phasor for the circuit current can be found easily by dividing the applied voltage’s phasor by the circuit’s impedance phasor,

\[ i = \frac{v_0}{z} = \frac{V_0}{Z}e^{i\varphi_z} = \frac{V_0}{Z}e^{-i\varphi_z}. \]  \hspace{1cm} (10.29)

We have used the fact that the applied voltage has zero phase so that its phasor is real \((e^{i0} = 1)\) and equal to its magnitude. We must note immediately that the current’s phase has opposite sign to the impedance’s phase whenever the applied voltage’s phasor is real. Now we can immediately write down the circuit’s current,

\[ I(t) = I_0 \sin(\omega t + \varphi_I) \]  \hspace{1cm} (10.30)

where

\[ I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \]  \hspace{1cm} (10.31)

and

\[ \varphi_I = -\varphi_z = \tan^{-1}\left(\frac{1}{\omega C} - \frac{\omega L}{R}\right). \]  \hspace{1cm} (10.32)

We can put Equation (10.31) into the standard form. First we bring out the resistance,

\[ I_0 = \frac{\frac{V_0}{R}}{\sqrt{1 + \left(\frac{Q}{\omega R}\right)^2 \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{\frac{V_0}{R}}{\sqrt{1 + Q^2 \left(\frac{R}{\omega_0 LC}\right)^2 \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{\frac{V_0}{R}}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_0} - \frac{\omega}{\omega_0}\right)^2} \left(\omega L - \frac{1}{\omega C}\right)^2}} \]  \hspace{1cm} (10.33)

and use the definitions of the resonance frequency, \(\omega_0\), and quality factor, \(Q\). We can also put Equation (10.32) into the standard form by using the definitions of quality factor and resonance frequency,

\[ \varphi_I = \tan^{-1}\left(\frac{\frac{1}{\omega C} - \omega L}{R}\right) = \tan^{-1}\left(\frac{QR}{\omega_0 L} - \frac{\omega L}{\omega_0}\right) = \tan^{-1}\left(\frac{\frac{1}{\omega_0 LC} - \omega}{\omega_L}\right) \]

\[ = \tan^{-1}\left(\frac{\frac{\omega_0^2}{\omega} - \frac{\omega^2}{\omega_0}}{\omega_0 \omega}\right) = \tan^{-1}\left(\frac{\omega_0^2 - \omega^2}{\omega_0 \omega}\right). \]  \hspace{1cm} (10.34)
Helpful Tip

We see immediately that resonances are completely determined by two parameters: the resonance frequency and the quality factor (or equivalently the bandwidth).

10.6.2 The Bandwidth

If we consider the power delivered to the circuit’s resistance, \( P = I^2R \), we will find that

\[
I^2(\omega) = \frac{I_0^2}{1 + \left[ Q \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right]^2}
\]

has a Lorentzian line shape. We define the bandwidth of this circuit response to be the full width at half maximum (FWHM) of the power that it delivers to its load, the resistance in this case. At \( \omega = \omega_0 \), the denominator of Equation (10.35) is 1; the power will be half as big when the denominator is 2. To make this development more convenient, we define \( w = \omega/\omega_0 \) so that

\[
2 = 1 + \left[ Q \left( w - \frac{1}{w} \right) \right]^2
\]

at the band edges. Solving for \( w \), we find that

\[
1 = \left[ Q \left( w - \frac{1}{w} \right) \right]^2
\]

\[
\pm 1 = Q \left( w - \frac{1}{w} \right)
\]

\[
0 = Qw^2 \pm w - Q
\]

\[
w = \frac{\pm 1 \pm \sqrt{1 + 4Q^2}}{2Q}.
\]

The radical is bigger than 1 so, to get positive frequencies, we must choose the positive radical. Then the bandwidth will be the range of frequencies between these two band edges, or

\[
\frac{\Delta f}{f_0} = \frac{\Delta \omega}{\omega_0} = \Delta w = \frac{1 + \sqrt{1 + 4Q^2}}{2Q} - \frac{-1 + \sqrt{1 + 4Q^2}}{2Q} = \frac{1}{Q} \text{ or } Q = \frac{f_0}{\Delta f}.
\]

Once again we see that larger values of \( Q \) (smaller resistances) result in a narrower passband, \( \Delta f \).
CHAPTER 10: EXPERIMENT 8

Checkpoint

What is meant by the response curve of a resonance in this experiment? All inductors have some resistance. Does this limit our ability to design a RLC circuit to respond only to an arbitrarily narrow frequency range of input signal? Why or why not?

If we recall correctly from Experiment 7, we should have noticed that suddenly changing the applied voltage (using a square wave in that case) caused the circuit to oscillate until circuit losses absorbed the energy in the oscillations. This circuit response has very important consequences for the communications industry. We observed that larger circuit resistance (or other losses) caused those ‘transient responses’ to decay more quickly. Having a circuit that can respond quickly allows the circuit to process information more quickly as well. However, today we learned that the bandwidth also depends upon losses

$$\Delta \omega = \frac{\omega_0}{Q} = \frac{\omega_0 R}{\omega_0 L} = \frac{R}{L}$$

so to process more information, we must also increase the width of the circuit’s passband. Although we have invented several other kinds of filter circuits to separate a wanted channel from hundreds of unwanted channels, we have not been able to get around this fundamental fact. Evidently a higher information rate simultaneously always requires a wider passband.

10.6.3 Logarithmic Unit Scales

Many times in science we discuss quantities on a logarithmic scale. Logarithmic scales must always be defined relative to a chosen reference, \( M = 10 \log_{10} \left( \frac{m}{m_0} \right) \). The units of logarithmic scales are deci-Bels (dB) after Alexander Graham Bell; note that 10 deci = 1. In the case of resonance, the reference power is the power at resonance, \( \omega = \omega_0 \), so that on the dB scale the power is

$$P_{\text{dB}} = 10 \log_{10} \left( \frac{P(\omega)}{P(\omega_0)} \right) = 10 \log_{10} \left( \frac{I^2(\omega)}{I_0^2} \right) = 20 \log_{10} \left( \frac{I(\omega)}{I_0} \right)$$

At the band edges, \( \frac{I^2}{I_0^2} = \frac{1}{2} \) and \( 10 \log_{10}(\frac{1}{2}) = -3 \text{ dB} \), so we frequently refer to the “-3dB points” to represent the band edges. Also at the band edges

$$\frac{I^2(\omega)}{I_0^2} = \frac{1}{2} \quad \text{so} \quad \frac{I(\omega)}{I_0} = \frac{1}{\sqrt{2}} \approx 0.707.$$

When the power is down to half, the current is only down to 70.7%. We can use this fact to get an estimate of the bandwidth very quickly.
10.6.4 Phase at the Band Edges

We also wish to discuss how the phase shift predicted by Equation (10.32) varies across the passband. It is easiest to see that at resonance the phase shift is \( \phi = 0 \). At the lower band edge,

\[
\omega_- \approx \omega_0 - \frac{\Delta \omega}{2} = \omega_0 - \frac{\omega_0}{2Q} = \frac{2Q - 1}{2Q} \omega_0
\]

\[
Q \frac{\omega_-^2 - \omega_0^2}{\omega_0 \omega_-} = Q \left( \frac{\omega_0}{\omega_-} - \frac{\omega_-}{\omega_0} \right) = Q \left( \frac{2Q}{2Q - 1} - \frac{2Q - 1}{2Q} \right) = \frac{4Q - 1}{4Q - 2} \approx 1
\]

\[
\phi_- \approx \tan^{-1} 1 = \frac{\pi}{4} = 45^\circ.
\]

In this case the argument of the response’s sine function reaches each value (2\( \pi \) for example) before the argument of the stimulus’ sine function. Similarly, \( \omega_+ \approx \omega_0 + \frac{\Delta \omega}{2} \),

\[
Q \frac{\omega_0^2 - \omega_+^2}{\omega_0 \omega_+} = \frac{-4Q + 1}{4Q + 2} \approx -1,
\]

and \( \phi_+ \approx -\frac{\pi}{4} = -45^\circ \). In this case the argument of the response’s sine function is always \( \pi/4 \) less than the argument of the stimulus’ sine function. Time must therefore become larger by 1/8 period to make up this difference before the current will reach the same phase as the stimulus’ phase has now.
Appendix A

Physical Units

In science, we describe processes in Nature using mathematics. Math is very concise, structured, and adaptable so that once we have mathematical models that mimic Nature, we can make very specific and accurate predictions about what will happen in other situations. Engineers use this fact to very good effect to obtain useful results in society.

Physical laws are usually expressed in terms of physical quantities that have dimensions. The definition of units that are used to describe these dimensions is the raison d’être of the National Institute of Standards and Technology (NIST). An example that we all should have seen by now is Newton’s second law,

\[ F(a) = ma \] (A.1)

Given a mass whose “value” is \( m \), to get motion whose acceleration is \( a \) we must apply a push or a pull whose “value” is \( F \). Since a push or a pull is an entirely different entity than a mass, we include in \( F \) a different multiplicative variable (N, dyne, pounds, etc.) than we include in \( m \)'s value (kg, g, slugs, etc.). The specific point that we need to address now is “What about the parameter \( a \)?” What intrinsic multiplicative variable should we include in \( a \)’s “value” so that the model is mathematically self-consistent? We can use algebra to figure this out.

Let the unknown units be \( x \) and consider the specific case of \( F = 1 \text{ N} \) and \( m = 1 \text{ kg} \). To solve our problem we substitute these values into the model and solve for our unknown, \( x \),

\[ 1 \text{ N} = F = ma = (1 \text{ kg})(1 x) \quad \text{so} \quad x = \frac{N}{\text{kg}} \] (A.2)

If we multiply all of \( a \)'s pure numbers by the units N/kg , then the unknowns (and unknowables) that we call units will always cancel in such a way that forces will always have units N, masses will always have units kg, and accelerations will always have units N/kg. Because all of these are always true, our model is mathematically self-consistent. Of course, kinematics has taught us that N/kg = m/s\(^2\) and that we can also determine acceleration by measuring the distances that objects move, the rates that objects move, and the times needed for these motions and rates to occur. One of the most beautiful and compelling aspects of physics is the frequency that a single physical entity (acceleration in this case) can be determined in
more than one way (sometimes many, many more than one).

Let us now move on to a more general case. Let us pretend that we have developed a
model represented by

\[ f(x) = ax + bx^3 \]  

(A.3)

Without more information the units of \( f, x, a, \) and \( b \) are ambiguous; so, let us stipulate that \( f \) is force (N) and that \( x \) is position (m). Is this enough information for us to determine the units of \( a \) and \( b \)? Our first instinct might be to answer that “No, one equation cannot yield two unknowns”; however, the rules of algebra must be obeyed as well. When we add the two quantities \( ax \) and \( bx^3 \), they must have the same units. Furthermore, these common units will be assigned to \( f \), so the results of these multiplications had better yield units of N! Now our problem has reduced to exactly the same problem as we solved above; we just have to solve it twice in the present case. Let \( A \) be the units of \( a \), let \( B \) be the units of \( b \), and substitute into the model:

\[ 1 \text{N} = f = ax = (1A)(1m) \quad \text{so} \quad A = \frac{\text{N}}{\text{m}} = \frac{\text{kg}}{\text{s}^2} \]  

(A.4)

\[ 1 \text{N} = f = bx^3 = (1B)(1m)^3 \quad \text{so} \quad B = \frac{\text{N}}{\text{m}^3} = \frac{\text{kg}}{\text{s}^2 \text{m}^2} \]  

(A.5)

Models having any number of terms can be handled in exactly this same way; we just have to solve the problem once for each term as above. Most students will also soon find short-cuts that will make this much more expedient than it may seem just now.

**Converting units**

Before looking at more complex situations, we will summarize how to convert between different physical units. Over the years many, many systems of units have been conceived and used for specifying distance. Historically, it has been important for economic, political, legal, personal, and scientific reasons that we have the ability to specify distance concisely and accurately. From a need to know our height or shoe size to a need to know the area of a land tract or to know how far away is the center of government, we need the ability to measure and to communicate distances. The problem is that different purposes benefit from somewhat different units of measure. It does not make sense to measure the distance to Los Angeles in human hair-widths, for example, even though you often hear small things being referred to “the thickness of a human hair” (~100 micrometers, by the way). To compare ‘apples’ to ‘apples’, we invariably have to convert between different units of measure.

The simplest strategy to convert between systems of units is to place the unit for distance in one system beside the unit for distance in the other system and simply to see how they compare. We use one of the “yardsticks” to measure the length of the other “yardstick”. As an example, we might use a meter stick to measure the length of a foot long ruler; when we do so we will find that 1 ft = 0.3048 m. We could also use the foot ruler to measure the meter stick and to find that 1 m = 3.2808 ft. Of course, these numbers are inverses of each
other, which is required for the measurements to be consistent. This implies that

\[ 0.3048 \, \frac{\text{m}}{\text{ft}} = 1 = 3.2808 \, \frac{\text{ft}}{\text{m}} \]  

\[ (A.6) \]

Let us suppose that we are given some distance \( d = 33.24 \, \text{ft} \) and that we need to communicate this distance to Paris, France, where no one has heard of “ft”; over there everyone measures distance in “m”. We can multiply anything by 1 without changing its value. This applies to \( d \) as well. We see that 1 = 0.3048 m/ft so that multiplying by 0.3048 m/ft is just multiplying by 1.

\[ d = 33.24 \, \text{ft} = 33.24 \, \text{ft} \times 1 = 33.24 \, \text{ft} \times 0.3048 \, \frac{\text{m}}{\text{ft}} = 10.13 \, \text{m} \]  

\[ (A.7) \]

**Dimensionless Functions**

Frequently we find expressions like

\[ f(x) = f_0 e^{ax} \text{ or } g(x) = g_0 \ln \left( \frac{x}{x_0} \right) \]  

\[ (A.8) \]

These functions belong to the class of exponential functions (recall that \( y = e^x \) if and only if \( x = \ln y \)). What must the units of \( f_0, a, g_0, \) and \( x_0 \) be in order that \( f \) and \( g \) be force (N) and that \( x \) be distance (m)? First, realize that \( e \approx 2.7183 \) is just a number, no different than 3, 8, or 1 as far as units are concerned. Multiplying pure numbers together does not (and cannot) cause units to appear; therefore, raising \( e \) to a power (multiplying \( e \) by itself over and over) also results in a pure number without units. Then \( f_0 \) and \( g_0 \) must have the same units as \( f \) and \( g \), respectively.

An exponent necessarily has no units. The only way to convert between units is to multiply by ratios of units and only raising to higher exponents \((x^a)^b = x^{ab}\) yields products of exponents. Units are algebraic unknowns so that having units in an exponent is equivalent to having the exponent to be unknown... not to know how many times to multiply the base. We might invent a rule or a convention simply to ignore the units – simply pretend that they are not there – just raise the base to whatever number is there. But that would mean that

\[ 1.6332 = 5^{0.3048 \, \text{m}} = 5^{1.000 \, \text{ft}} = 5 \]  

\[ (A.9) \]

which is clearly not true. On the other hand, requiring that exponents have no units works out a little better. To show this we re-arrange our english-metric conversions a little

\[ 1 = 3.2808 \, \frac{\text{ft}}{\text{m}} \text{ and } \frac{\text{ft}}{\text{m}} = 0.3048 \]  

\[ (A.10) \]

and manipulate them similar to the way we did above. Let \( b \) be some arbitrary number and note that

\[ b = b^1 = b^{3.2808 \, \frac{\text{m}}{\text{ft}}} = b^{(3.2808)(0.3048)} = b^1 \]  

\[ (A.11) \]
which is self-consistent.

Exponents must be pure numbers. For similar reasons the arguments of logarithms must also be unitless, pure numbers. To see this recall that

\[ \ln \left( \frac{x}{x_0} \right) = \ln(x) - \ln(x_0) \quad (A.12) \]

If \( x \) and \( x_0 \) have the same units, the units cancel on the left and leave a pure number to be the logarithm’s argument. If they have different units, then any unit conversion applied to \( x_0 \) on the right would change the value subtracted from \( \ln(x) \) thus implying that the quantity on the left can simultaneously be equal to both values. So, how do we deal with the right side of Equation (A.12) if \( x \) and \( x_0 \) have units? We understand that we actually mean something different:

\[ \ln(x) - \ln(x_0) = \ln(x) - \ln(x_0) + \ln(a) - \ln(a) = \ln(x/a) - \ln(x_0/a) = \ln \left( \frac{x}{x_0} \right) \quad (A.13) \]

The constant \( a \) must have the same dimensions as \( x \) and \( x_0 \), and although it cancels entirely from the expression, it must be there to make the logarithm’s dimensionless argument make sense. Incidentally, this is also why logarithmic scales like the Richter scale for earthquakes and the dB scale for sound always have a reference intensity (a definite value with units), equivalent to \( x_0 \) or \( a \) in the discussion above.

Sine, cosine, and tangent are trigonometric functions whose arguments are usually derived from angles. Trigonometric functions arguments are dimensionless, but they are not unitless; these arguments must have the units of their angular measure. What this means is that if you scale physical dimensions, the angles do not change...if you change your angular measure, then the angle’s numerical value (degrees or radians) does change. Frequently, we deal with expressions like

\[ x(t) = x_0 \cos(\omega t + \phi) \quad (A.14) \]

If \( x \) has units of m and \( t \) has units of s, what are the units of \( x_0 \), \( \omega \), and \( \phi \)? The cosine function itself has no units so \( x_0 \) has the units of \( x \). The argument of the cosine is a polynomial, so each term must have the angular units that the cosine function needs. The natural units of angle are radians so that a circle has \( 2\pi \) radians. The units of \( \omega \) must be rad/s. You might notice a similarity between logarithms, exponentials, and trigonometric functions: their arguments are all dimensionless. This is not an accident. Mathematics typically concerns functions applied to numbers. Thus, mathematical expressions tend to be defined in a way such that they have no dimensions. It is only in physics, where we have to compare real measurements of different quantities, where the reference units appear. Therefore, in these mathematical functions, we must convert the argument to pure numbers for the expressions to make sense.
Appendix B

Using Vernier Graphical Analysis

Additional guidance is generally provided in the lab experiment chapters.

**Helpful Tip**

There are two versions of this software on the lab computers. The older one, ‘Gax’ is not well-liked and has caused many TA and student headaches throughout the years. The newer version, ‘Graphical Analysis 3.4’ (Ga3) is more functional and reliable but requires a few more steps to get it to do what you need it to do. Please use Graphical Analysis 3.4 in this course. Excel and Gax are always available if you prefer, however.

When we perform a least squares fit to a mathematical model, we are effectively compiling all of the knowledge gained by our data into a much smaller set of fitting parameters. To the extent that our data applies to this particular model, the experimental errors that are inevitable in measurements are averaged out so that we might utilize the fitting parameters to improve upon a prediction of our model beyond the data points themselves. This does not mean that the fitting parameters have no uncertainties associated with them; indeed they do have and it is essential that we specify each uncertainty when we discuss the parameter.

Many times Graphical Analysis 3.4 does not show the uncertainties in fitting parameters by default. We can always ask it to do so by right-clicking the parameters box, “Properties...”, and select “Show Uncertainties”. We can also specify other properties in the fitting parameters such as the number of significant digits.

Not only is the older Gax somewhat unstable, but it also offers meaningful estimates of fitting parameter uncertainties *only* for the separate ‘linear regression’; custom models have no convenient way to get uncertainty estimates. Since fitting parameters are effectively measurements that the computer has deduced from the data, it is essential that we be able to specify the uncertainties and the units that must accompany them. These are the reasons why we prefer Ga3 that has a less obvious user interface.
Appendix C

Using Microsoft Excel

C.1 Creating plots and curve fits

Helpful Tip

The primary advantage of Excel is that students have access to it at home. However, it is more challenging to perform statistical-based fits in Excel unless you write your own spreadsheet or macros. It is recommended that you perform your data analysis with Graphical Analysis 3.4 in the lab. However, you always have access to Excel in many locations in case you want to plot outside the lab.

This short list of resources will help guide you in creating figures in Excel, if you decide to use it.

- A good introduction to basic graphing in Excel is provided by the LabWrite project: https://www.ncsu.edu/labwrite/res/gt/graphtut-home.html

- Curve fitting in Excel can be performed, but sometimes it is much more painful than using a dedicated data analysis program. For least-squares curve fitting, this link details how to use Excel:
  http://www.jkp-ads.com/articles/leastsquares.asp

  Neither do Excel’s least-squares fitting parameters come specified with uncertainty estimates even after you go through these acrobatics.

- We can use Excel’s “LINEST” function to provide complete statistics on straight lines. See ‘linest’ in Excel’s Help files to learn more about its usage.
C.2 Performing Calculations

All calculations in spreadsheets begin with “=” First, select the cell that will hold the result of the calculation, type “=”, and enter the formula to be evaluated Typing “=5+9” and ‘Enter’ will yield “14” in the cell; the cell beneath will then be selected. If you need a cell to contain ‘=’ instead the results of a calculation, precede it with a single quote ‘’...anything you type after that will be displayed verbatim. Even numeric characters won’t be numbers after ‘’.

If this was the extent of their capabilities, spreadsheets would not be very popular. Each spreadsheet cell is at the intersection of a column having alphabetic label at the top and a row having numeric label at the left. ‘A1’ is the cell at the top left corner of the sheet, ‘C8’ is at the eight row and third column, and ‘Z26’ is at the 26th row and the 26th column. After column ‘Z’ comes column ‘AA’ to ‘AZ’, ‘BA’ to ‘BZ’,..., ‘ZA’ to ‘ZZ’. If even more columns are needed, three, four, and five characters can be used. Column ‘numbers’ count up just like decimal digits, but column designations have base 26 instead of base 10.

As an exercise we could put the numbers 1-10 in cells A1:A10. Then in, say, B7 we could type ‘=A1+A2+A3+A4+A5+A6+A7+A8+A9+A10’ to add the numbers (55). After we type ‘=’, we could click on cell A1 to get the computer to place the ‘A1’ in the formula. This comes in handy in a large project when you don’t remember the cell designation and it is not visible on the screen.

This is pretty nice and the total (cell B7) will automatically update anytime one or more of the numbers in A1:A10 changes. However, this would mean a lot of typing if the sum of, say, A1:A1000 were needed instead. It turns out that spreadsheets have a ‘sum’ function for adding the contents of cells. “=sum(A1:A10)” would also add the contents of these cells. ANY block of cells can be specified using “range” designators. “=sum(a1:z6)” will add the first six rows of the first 26 columns. The function names and column designators are not sensitive to case. The cell into which the result is to be placed MUST NOT be part of the range or an error will result. The ‘A1:A10’ or ‘A1:Z6’ can be generated by the computer by dragging the mouse across the desired range of cells.

In addition to ‘sum’, we could ‘average’, ‘stdev’, ‘count’, etc. to perform statistics on a range. Many other functions are part of the standard spreadsheet application. A partitioned list and usage instructions can be generated using the ‘function’ toolbar button; usually the icon has ‘f’, ‘fx’, ‘f(x)’, or such.

Frequently, we want to generate graphs. To do this we first need to generate the independent (x-axis) values. Enter ‘10’ into say C3 and ‘=0.1+C3’ into C4. Now, go back to C4 and ctrl+c to copy this formula. Next, drag the mouse from C5 to C103 to select this range (the cells turn black) and ctrl+v to paste the formula into all of the selected cells. Now, C3=10, C4=10.1,..., C103=20. Click on one of these cells and look at the formula in the formula bar. The computer has automatically replaced the ‘3’ in ‘C3’ with the correct number to add 0.1 to the cell above this one! This always happens by default when you copy and paste the formulas from one cell to another unless you “Edit/Paste Special...” and specify the details you want pasted.
For relatively smooth functions, these 101 values will yield a very nice graph. Smaller increments can also be generated at need for more quickly changing functions. Uniformly spaced numbers are needed so often that a shortcut exists. Type 10 into C3 and 10.1 into C4. Next, drag from C3 to C4 to select both and release the mouse button. The bottom right corner of the selected cells is a drag handle that can be used to extend the pattern 10.0, 10.1, 10.2, . . . by dragging this handle downward with the left mouse button. Rows of incrementing numbers can be generated similarly.

Now that we have our abscissa, we need to generate our ordinate. Click on D3 and enter ‘=3*sin(C3)’. Go back to D3 and drag the drag handle at the bottom right downward to D103. Release the mouse button and note that the result is a sinusoid with amplitude 3. This is OK! but we can generate families of curves with varying amplitudes, frequencies, or phases quite easily. Enter ‘2’ into D2, ‘2.2’ into E2, . . . , ‘3’ into I2. Now enter ‘=D$2*cos($C3)’ into D3. Go back to D3, drag the drag handle across to I3, and release the mouse button. Now, drag the drag handle down to I103 and note that we have generated six sinusoids with amplitudes given by row 2. The ‘$’ prevented the ‘2’ in ‘D2’ and the ‘C’ in ‘C3’ from changing. Using ‘$A$1’ or ‘$B$3’ would insert the number in A1 or B3 into all of the formulas; this would allow you to tweak these two numbers by hand to optimize all of the curves at once.

Practice generating various abscissas and ordinates so that you can remember these basics. Also practice plotting the results and pasting the plots into a Word document. You will want ‘scatter plots’ so that you can choose the \((x, y)\) points.
Using Microsoft Word

A nice template containing a report outline and hints for performing many of the functions science reports, journal publications, and books often employ can be downloaded from each lab’s website. All word processing programs have similar capabilities and many can import the template…although perhaps not seamlessly. Hopefully, this will get you started using word processing software.

For your electronic write-ups, you can embed figures directly in the Word files, or you can upload them separately. If you want to embed in Microsoft Word while preparing your document, here are some useful sources of assistance.

- To embed an Excel figure directly into Word, you can follow the instructions here:
  https://support.office.com/en-us/article/Insert-a-chart-from-an-Excel-spreadsheet-into-Word-0b4d40a5-3544-4dcd-b28f-ba82a9b9f1e1

- If you want to embed a PDF figure into a Word document, you can follow the instructions here:
  https://support.office.com/en-za/article/Add-a-PDF-to-a-document-9a293b43-45de-4ad2-a0b7-55a734cf6458

- Another set of instructions for embedding PDFs in Word, with a focus on preserving quality, can be found here:

- For embedding more general images in Word, and using word wrapping, see here:
  https://support.microsoft.com/en-us/kb/312799
With Microsoft Word, there are usually many ways to accomplish the same task. Some produce better results than others. This is why you always have the option of uploading figures and images separately from the text of your write-up.

**Helpful Tip**

Your goal for writing in this lab is clarity in communication, not professional quality documents. Beginning students have much difficulty including enough background information about their experiment and apparatus and they tend to include too much mundane detail that applies only to their specific apparatus instead. Eventually our work will be read by scientists all over the world, so it is essential that we practice including enough information about our apparatus and procedure that everyone in the world can understand our data while simultaneously avoiding these mundane details that will tend to confuse readers without our apparatus in front of them.
Appendix E

Composing a Report

E.1 Title and Partner Credit

Each report must have a relevant title. Each report must list all lab partners with the report author listed first. Technical reports begin at a general viewpoint discussing widely known science and angle toward more specific aspects of this experiment and the particular science that is within its scope.

E.2 Purpose or Introduction

We typically suggest 3–5 brief sentences that note the Purposes of the experiment. What physics will the experiment test? Usually physics is expressed as equations; however, frequently these equations are 'named'. The widely-known name is a preferred mode of address, but a forward citation to your Theory is second-best. (In this case Theory is absent so use the lab manual equation number.)

What new OEM instruments and devices will we learn to use? What physical quantities will we measure?

E.3 Theory

The Theory section would be quite similar to the mathematical development in this manual. Because of this, we do not require a Theory section in your reports. You may cite the equation numbers from this manual as if you had typed the relevant development into your report. In professional papers you will need to demonstrate that your apparatus and procedure are accurate representations of the science you wish to test; in these cases you may refer to similar developments in this manual for ideas in how to proceed.
E.4 Apparatus and Procedure

What specific instruments and devices were used? Manufacturer names and model numbers allow the reader to seek technical specifications directly from the manufacturer. How were the devices connected and/or utilized? What data did each device report? What system state did each device control? If the Data do not make it redundant, what Procedure was followed to generate the data? This section will answer these questions, but they can usually be answered in pairs and triplets by well structured sentences. Content is important, but length is not important.

Frequently, a sketch, a schematic diagram, or a photograph can convey much information and quickly. Students may copy and paste such illustrations from the lab manual, the website, etc.; however, he must cite the source in each case. Citations must lead the reader directly and completely to the cited object. A book name, publisher, and page number (or figure label) is a valid citation. A url and figure label is another. Furthermore, we can give a student permission to copy only the illustrations that we own. Copying from other sources is not allowed at all without prior permission from those authors.

E.5 Data and Results

Briefly state general observations and publish data tables and graphs to portray your measurements. Graphs usually will also include a fitted model of the physics equation(s). Frequently, there will also be ‘singleton’ measurements that do not logically fit into any of the tables. Sometimes a ‘miscellaneous’ measurements table can house these and other times these measurements are worked into paragraphs. The visibility and tidiness of tables and graphs make these favorite places to portray data.

All measurements have three pieces:

1. The best estimate $m$ for the measured value,

2. An objective estimate $\delta m$ of measurement uncertainty, and

3. Units (U) multiplying the best estimate and uncertainty.

No measurement is complete until all three pieces are included: $M = (m \pm \delta m) \text{U}$. Measurements in paragraphs are disclosed using this format. Measurements in tables have the name $(M)$ and units (U) in the table column headers and the best estimate and uncertainty $(m \pm \delta m)$ in the respective table column. Graphs have the best estimate represented as a location on the graph, the uncertainty represented by error bars, and the measurement name and units in the axis titles. We will not specify error bars, but we will repeat some of the graph’s measurements in suitable tables that do specify uncertainty. Professional reports will utilize only one of these formats for each data set; but students are not yet professionals.

As described in Section 2.5, the uncertainty tells us how well we know the measured value so we need only 1-2 significant digits in the uncertainty. Since the uncertainty does tell us how
well we know the measured value, it is necessary that these two numbers have exactly the same number of decimal places: (10.2 ± 0.1), (13.935 ± 0.025), and (1250 ± 10) are all correctly specified (except for absent units); however, (10.2 ± 0.05), (10.225 ± 0.2), and (2.2258 ± 0.1254) are all incorrect at this level.

Usually the raw measurements will be used in physics model equations to predict other measurements. Show one example of how you calculated each of these predictions from the raw data; we prefer that you show the formula with symbols, the numbers substituting for the symbols, and the result with correct units. In 2–3 experiments during the quarter, your TA will also specify that you use the methods in Section 2.6.3 to specify the uncertainties in these derived results; please show one example for each of these as well when required. Additionally, specify which table columns contain raw data and which contain derived results.

### E.6 Analysis of Results

In the Analysis section, we will decide whether our data prove anything and, if so, what they prove. First, do the model curves fit the data points? If so, this is strong evidence that the data supports the model even if some of the model parameters are different than expected.

To determine whether two numbers are distinguishable by the experiment, we rely heavily upon statistics to decide, but we also simply use the relevant statistics results without proof. The strategy (see Section 2.9.1) is to form NULL hypotheses by subtracting results computed using a model under test from a direct measurement taken to check the model

\[ \Delta M = |m_P - m_M|. \]  

We also need the acceptable error (\( \sigma \) or ‘sigma’) in this difference

\[ \sigma_M = \sqrt{(\delta m_P)^2 + (\delta m_M)^2}. \]  

### General Information

The following observations are also discussed in Section 2.9.1.

Statistics tells us that 68.3% of the time \( \Delta M < \sigma_M \) if \( M_P \) and \( M_M \) are taken from the same distribution. Similarly, 95.4% of the time \( \Delta M < 2\sigma_M \) and 99.73% of the time \( \Delta M < 3\sigma_M \). These probabilities get progressively more certain as agreement requirements get poorer. We may also turn this argument around. If we stated that \( \Delta M > \sigma_M \) means that the prediction and measurement are different, then this statement would be wrong 32% of the time (about 1/3 of the time). Insisting that \( \Delta M > 2\sigma_M \) means the numbers are different would make us wrong 4.6% of the time (about 1/22 of the time). Having \( \Delta M > 3\sigma_M \) happens due to statistics alone only 0.27% of the time (about 1/370 of the time).
Being wrong 1/3 of the time is considered unsavory among scientists. Being wrong 1/22 of the time, even, is undesired. Thus we generally expect and state that when \( \Delta M < 2\sigma_M \) the two numbers probably do come from the same distribution and any disagreement is due to “other sources of error.” Contrarily, having \( \Delta M > 3\sigma_M \) happens only 1/370 of the time so we usually expect that this difference probably has a non-statistical cause; we reject the hypothesis that these two numbers are taken from the same distribution. This does not necessarily mean that the physics model is incorrect, but this does remain one possibility. Other possibilities include 1) one or more of the assumptions needed for the model to be valid are not met, 2) we have underestimated (or completely overlooked) some of our uncertainties, 3) the environment has perturbed the experiment, 4) the experiment’s performance was substandard, etc.

We also need to enumerate several of the most likely “other sources of error” that might cause any observed discrepancies however small. Our considerations of how well the model curves fit the data points, of how well the parameters match what we should expect, of how well other observations match the model’s predictions are all relevant to the conclusions we draw.

E.7 Conclusions

This section should be very brief, but it also should be self-reliant. Which of our original hypotheses do our data support? Which do they contradict? Which are neither supported nor contradicted? We do not say why; (everything above says why.) Equations may be referenced via citation or accepted name (Newton’s law, conservation of energy, Equation (4), etc.). What physical measurements have you made? These are constants independent of the experiment that we (or others) might need one day. Can you think of any applications for what we have observed? Finally, how might we improve the experiment if we should perform it again?
Appendix F

Submitting a Report in Canvas

Preparation

Your lab report should be written in either Microsoft Word or be converted to a text-readable pdf. Every student at Northwestern has access to Microsoft Word and Excel for lab report preparation. JPEG format is not allowed for the main text.

Both Word and pdf formats allow your TA to provide helpful feedback using Canvas’ SpeedGrader system. Experience has shown that this online feedback is popular to students in the 136 labs. This formatting requirement is therefore beneficial to everyone involved.

If possible, it is easiest if all of your files are merged into a single document. In Word, images and figures can be embedded directly into the text file. If you are submitting a pdf prepared in another method, it is possible to attach images and figures saved in the lab as pdf documents into a single pdf file. Adobe Acrobat Professional can accomplish this. Alternatively, you can merge pdf files online. One possible provider is PDFMerge (http://www.pdfmerge.com/).

If you cannot put all of your content into a single file, it is accessible to load figures as separate files (such as jpeg files). If you submit as separate files, you should refer to each uploaded figure in some other part of your Lab Report. Your TA is not required to view or grade any document that is not referred to in other graded sections of your report. It is recommended that you name your files “Figure X” where X is a number. Then you can refer to each figure by name. Alternatively, you can upload figures as a single document such as a pdf, and have the appropriate labels on the figures in each document. Remember: a lab report is practice in communication of ideas and activities. Failure to make references to uploaded figures clear to your TA can result in lost points.

Submission

To submit your lab report, open the assignment in the Module on Canvas. Click the “Submit Assignment” link.
Under the File Upload tab, select the “Choose File” button and select the appropriate file. You can press “Add Another File” to create another upload link. When all files are ready and you agree to the TurnItIn pledge, you can press “Submit Assignment”. You are able to re-submit your assignment to Canvas, although if it is late it will be flagged. You should also be able to view your TurnItIn similarity score a few minutes after submission.

Your TA will be able to see all submitted files in the Canvas grading system. It is probably easiest on your TA if the first file that you upload is your main lab report text, followed by figure files in order. Do not make life harder for your TAs as they grade your work!